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Professor A.A. Martynyuk
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Method of Lines to Hyperbolic Integro-Differential Equations in $\mathbb{R}^n$

D. Bahuguna [1*], J. Dabas [1] and R.K. Shukla [2]

[1] *Department of Mathematics*
Indian Institute of Technology, Kanpur - 208 016, India.
[2] *Invertis Institute of Engineering Technology*
Bareilly, India

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Abstract: In this work we consider a hyperbolic integro-differential equation in $\mathbb{R}^n$. We reformulate it into an evolution equation in a suitable Hilbert space and establish the existence and uniqueness of a strong solution using the method of lines and the theory of semigroups of contractions in a Hilbert space.

Keywords: *Wave equation; semigroups; contractions; method of lines; strong solution.*

Mathematics Subject Classification (2000): 34G20, 35L90,35L15, 47D03.

1 Introduction

In this paper we are concerned with the following perturbed wave equation in $\mathbb{R}^n$,

$$\begin{cases} \frac{\partial^2 w}{\partial t^2}(x,t) - \Delta w(x,t) = f(x,t) + \int_0^t k(t-s)\Delta w(x,s)\, ds, \ (x,t) \in \mathbb{R}^n \times (0,T], \\ w(x,0) = f_1(x), \quad \frac{\partial w}{\partial t}(x,0) = f_2(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where Δ denotes the n-dimensional Laplacian, the unknown real valued function w is to be defined on $\mathbb{R}^n \times [0,T]$, $0 < T < \infty$, k is a real valued function defined on $[0,T]$, the real valued function f is defined on $\mathbb{R}^n \times [0,T]$, the real valued functions f_i are defined on $\mathbb{R}^n$, $i = 1, 2$.

The problem (1) with $k \equiv 0$ has been extensively treated by many authors, see, for instance, Yosida [21, 22] and Pazy [18]. Our aim is to reformulate (1) as a first order

* Corresponding author: dhiren@iitk.ac.in

evolution integro-differential equation in a suitable product Hilbert space and apply the theory of semigroups together with the method of time discretization in time to establish the existence and uniqueness of solutions.

One may formulate (1) as a second order evolution equation in a Hilbert space. Goldstein [13] has used semigroup of bounded linear operators to second order evolution equations. But we use the ideas of Pazy [18] to put (1) in a product Hilbert space taken as a Cartesian product of two Hilbert spaces. It turns out that the operator associated with the differentiation is an infinitesimal generator of a group of contractions in the chosen product Hilbert space. We incorporate the properties of such operators with the method of lines to establish the existence of a strong solution.

As a model for the foregoing equation we consider equations of the form:

$$u_t(x,t) = (k(x)u_x(x,t))_x + \int_0^t G(t,s)(\sigma(u_x(x,s)))_x ds + h(x,t). \tag{2}$$

Such equations have physical application; for example, they arise in problems concerned with heat flow in materials with memory. Linear versions of equation (2) are treated in [16] and [17]; the nonlinear versions are treated in [15] and similar equations are also treated in [14]. If we replace $u_t(x,t)$ by $u_{tt}(x,t)$ we obtain an equation arising in the theory of viscoelasticity [8]. Our results for the evolution integro-differential equation may also be applied to the heat conduction problem for a material with memory (cf. Liu and Ezzinbi [12]),

$$u_t(x,t) = u_{xx}(x,t) + \int_0^t k(t-s)u_{xx}(x,s)\, ds + f(x,t), \quad t > 0, \quad x \in (a,b), \tag{3}$$

$$u(x,0) = u_0(x). \tag{4}$$

The above problem can be teated as a particular case of our study even in the case when $(a,b) = \mathbb{R}$. whereas in [12] it is a bounded interval only. In [7] the authors study the following functional integro-differential equation in the product Hilbert space $\mathcal{H} := H_0^1(0,1) \times L^2(0,1)$,

$$\begin{cases} \frac{du}{dt} - Au = \int_0^t k(t,s)Au(s)ds + F(t,u_t), \\ u_0 = \phi, \end{cases} \tag{5}$$

where $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is shown to be the infinitesimal generator of a contraction semigroup in $\mathcal{H}$ and the nonlinear function $F : [0,T] \times \mathcal{C}_0 \to \mathcal{H}$. Here the space $\mathcal{C}_t := C([-T,t];\mathcal{H})$, $t \in [0,T]$, is the Banach space of all continuous functions from $[-T,t]$ into $\mathcal{H}$ endowed with the supremum norm. By the application of the method of lines the existence and uniqueness of a strong solution is proved.

Our aim is to apply Rothe's method to establish the existence and uniqueness of a strong solution which in turn will guarantee the well-posedness of (1). The method of lines is a powerful tool for proving the existence and uniqueness of solutions to evolution equations. This method is oriented towards the numerical approximations. For instance, we refer to Rektorys [19] for a rich illustration of the method applied to various interesting physical problems. For the application of the method of lines to nonlinear differential and Volterra integro-differential equations (VIDEs) in which bounded, though nonlinear, operators appear inside the integrals, see Kacur [9, 10], Rektorys [19], Bahuguna and Raghavendra [4]. Recently, Bahuguna and Dabas [6] have considered a nonlocal problem arising in the population dynamics using Rothe's Method. In the present study we extend the application of the method of lines to a class of nonlinear VIDEs in which differential operators occur inside the integrals and hence are unbounded. Motivation for considering such problems arises from the theory of wave propagation under the influence of damping, see Bahuguna [2], and Bahuguna and Shukla [3] and references cited therein.

2 Preliminaries

Here we briefly describe the spaces required in subsequent analysis. For details we refer to Pazy [18]. Let $\Omega \subset \mathbb{R}^n$ be a domain with sufficiently smooth boundary $\partial\Omega$ and $\overline{\Omega}$ be the closure of Ω in $\mathbb{R}^n$. Let $m \in \mathbb{N} \cup \{0\}$ and let $C^m(\Omega)$ ($C^m(\overline{\Omega})$) be the set of all m-times continuously differentiable real valued functions on Ω ($\overline{\Omega}$). Let $C_0^m(\Omega)$ be the subset of all functions in $C^m(\Omega)$ having compact support in Ω. For $1 \leq p < \infty$, we define a norm in $C^m(\Omega)$ by

$$\|u\|_{m,p}^p = \int_\Omega \sum_{|\alpha| \leq m} |D^\alpha u(x)|^p \, dx, \quad u \in C^m(\Omega),$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, n$, is a multi-index with $|\alpha| = \sum_{i=1}^n \alpha_i$ and D^α is the partial differential operator

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots x_n^{\alpha_n}}.$$

Let $C^{m,p}(\Omega)$ be the subset of all $u \in C^m(\Omega)$ such that $\|u\|_{m,p} < \infty$. For $p = 2$ we also define the inner product

$$(u, v)_m = \int_\Omega \sum_{|\alpha| \leq m} (D^\alpha u)(D^\alpha v) \, dx.$$

The Banach spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are defined as the completion of $C^{m,p}(\Omega)$ and $C_0^{m,p}(\Omega)$ with respect to the norm $\|.\|_{m,p}$, respectively. For $p = 2$, we denote the Hilbert spaces $W^{m,2}(\Omega)$ and $W_0^{m,2}(\Omega)$ as $H^m(\Omega)$ and $H_0^m(\Omega)$, respectively. For $\Omega = \mathbb{R}^n$, we write $H^m = H^m(\mathbb{R}^n)$. If given function w defined on $\mathbb{R}^n \times [0, T]$ into $\mathbb{R}$ such that for each $t \in [0, T]$, $w(., t) \in H^0$, then we may identify w from $[0, T]$ into H^0 by $w(t)(x) = w(x, t)$, $x \in \mathbb{R}^n$. In addition, if $\frac{\partial w}{\partial t}(., t) \in H^0$ for each $t \in [0, T]$ then $\frac{dw}{dt}$ is also defined as a function from $[0, T]$ into H^0. We now consider the product Hilbert space $H = H^1 \times H^0$ which is the completion of $C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\begin{aligned} |||u|||^2 &= |||(u_1, u_2)|||^2 = \left(\int_{\mathbb{R}^n} (|u_1(x)|^2 + |\nabla u_1(x)|^2 + |u_2(x)|^2) \, dx \right), \\ & (u_1, u_2) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n). \end{aligned} \tag{6}$$

The Hilbert space H^m may also be characterized as the space of all functions $f \in$ such that the Fourier transform $\hat{f}$ has the property that $(1 + |\xi|^2)^{k/2} \hat{f}(\xi)$ is in H^0 as a function of $\xi \in \mathbb{R}^n$.

In order to write (1) as an evolution equation in H we take $u = (u_1, u_2)$ where $u_1 = w$ and $u_2 = \frac{\partial w}{\partial t}$. We define the operator $A : D(A) \subset H \to H$ as $D(A) = H^2 \times H^1$ with $Au = A(u_1, u_2) = (u_2, \Delta u_1) - 2(u_1, u_2)$.

Thus, the problem (1) is equivalently represented by the following evolution equation in H as

$$\frac{du(t)}{dt} = Au(t) + F(t, u(t)) + \int_0^t k(t - s) Bu(s) \, ds, \ t \in (0, T], \ u(0) = u_0, \tag{7}$$

where $D(B) = H^2 \times H^0$ with $Bu = B[u_1, u_2] = [0, \Delta u_1]$ for $u \in D(B)$ and $u_0 = (f_1, f_2)$, under the assumption that $(f_1, f_2) \in H^1 \times H^0$ and $F : [0, T] \times H \to H$ given by $F(t, u) = [0, 2u + f(t)]$, $f : [0, T] \to H^0$, $f(t)(x) = f(x, t)$.

3 Existence and Uniqueness of Solutions

In this section we continue to use the notations and notions introduced in the earlier sections and consider the well-posedness of (7).

We observe some of the properties of the operators A and B and assume Lipschitz continuity of the kernel k.

(P1) The operator $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0 of group $T(t)$ in H. (Cf. Pazy [18] Theorem 7.4.5, pp. 222–223.)

(P2) $D(A) \subset D(B)$ and $Bu = Au + Pu$ where $P : H^1 \times H^1 \to H$ is a bounded linear operator given by $Pu = P[u_1, u_2] = [-u_2, 0] + 2[u_1, u_2]$ with $|||Bu||| \leq C(|||Au||| + |||u|||)$ for $u \in D(A)$ and $|||Pu||| \leq C|||u|||$ for $u \in D(P)$.

(P3) The function $F : [0, T] \times H \to H$ satisfies

$$|||F(t, u) - F(s, v)||| \leq L_F[|t - s| + |||u - v|||],$$

for $t, s \in [0, T]$ and $u, v \in H$. The function $k : [0, T] \to \mathbb{R}$ is Lipschiz continuous with $k(0) = 0$.

By $C^{0,1}([0, T]; H)$ we denote the Banach space of all Lipschitz continuous functions from $[0, T]$ into H endowed with the norm

$$\|u\|_{C^{0,1}} = \max_{0 \leq t \leq T} |||u(t)||| + \sup \left\{ \frac{|||u(t) - u(s)|||}{|t - s|} : t, s \in [0, T], t \neq s \right\}.$$

We have the following main result.

Theorem 3.1 *If (P1), (P2) and (P3) are satisfied then for each $(f_1, f_2) \in H^2 \times H^1$, there exists a $u \in C^{0,1}([0, T]; H)$ with $u(0) = u_0$ satisfying (7) a.e. on $[0, T]$. Furthermore, if $k \in C^1[0, T]$, then the solution u is unique.*

4 Basic Lemmas

We shall prove Theorem 3.1 with the help of several lemmas. We devide the interval $[0, T]$ into subintervals $[t^l_{j-1}, t^l_j]$, $t^l_j = j.h$, $h = T/l$, $j = 0, 1, 2, \ldots, l$. We set $u^l_0 = u_0$ for all $l \in \mathbb{N}$. For $j = 1, 2, \ldots, l$, we define by u^l_j the unique solutions of each of the equations

$$\frac{u - u^l_{j-1}}{h} - Au = F^l_j + h \sum_{i=0}^{j-1} k^l_{ji}\ Bu^l_i, \tag{8}$$

where

$$F^l_j = F(t^l_j, u^l_{j-1}), \tag{9}$$
$$k^l_{ji} = k(t^l_j - t^l_i). \tag{10}$$

The existence of unique u^l_j satisfying (8) is ensured by Corollary 7.4.4 in Pazy [18] which is stated here as follows.

Lemma 4.1 *Let* $A_0 : (D(A_0) \subset H \to H$ *with* $D(A_0) = D(A)$ *and* $A_0[u_1, u_2] = [u_2, \Delta u_1]$ *Then for any* $v \in H$, *and* $0 < |\lambda| < 1/2$, *the equation*

$$u - \lambda A_0 u = v$$

has a unique solution $u \in D(A_0)$ *satisfying the estimate*

$$|||u||| \leq (1 - 2|\lambda|)^{-1} |||v|||.$$

Furthermore A_0 *is the infinitesimal generator of a group* $\{S(t) : t \in \mathbb{R}\}$ *of bounded linear operators in* H *with*

$$\|S(t)\|_{B(H)} \leq e^{2|t|},$$

where $B(H)$ *denotes the Banach space of bounded linear operators in* H *with the norm* $\|.\|_{B(H)}$.

Remark 4.1 From Lemma 4.1 it follows that A is the infinitesimal generator of the contraction group $\{T(t) : t \in \mathbb{R}\}$ of bounded linear operators in H where $T(t) = e^{-2|t|}S(t)$. Therefore by Lumer-Phillips Theorem (cf. Pazy [18]) that A is m-dissipative, i.e.,

$$(Au, u) \leq 0, \quad u \in D(A)$$

and $R(I - \lambda A) = H$ for all $\lambda > 0$.

In order to ensure unique solution $u_j^l \in D(A)$ of (8) with the help of Lemma 4.1 we rewrite it as

$$u - hAu = u_{j-1}^l + hF_j^l + h^2 \sum_{i=0}^{j-1} k_{ji}^l B u_i^l.$$

Now, the existence of a unique u_j^l satisfying (8) follows from the m-dissipativity of A.

Now, we show that $\delta u_j^l = (u_j^l - u_{j-1}^l)/h$ lie in a ball of fixed radius independent of the discretization parameters j, h and l. For convenience, we shall denote by C a generic constant, i.e., KC, e^{KC}, etc., will be replaced by C where K is a positive constant independent of j, h and l.

We shall use later the following lemma due to Sloan and Thomee [20].

Lemma 4.2 *Let* $\{w_l\}$ *be a sequence of nonnegative real numbers satisfying*

$$w_l \leq \alpha_l + \sum_{i=0}^{l-1} \beta_i w_i, \quad l > 0,$$

where $\{\alpha_l\}$ *is a nondecreasing sequence of nonnegative real numbers and* $\beta_l \geq 0$. *Then*

$$w_l \leq \alpha_l \exp\{\sum_{i=0}^{l-1} \beta_i\}, \quad l > 0.$$

Furthermore, we also require the following lemma for later use.

Lemma 4.3 *Let $C>0$, $h>0$ and let $\{\alpha_j\}_{j=1}^{l}$ be a sequence of nonnegative real numbers satisfying*

$$\alpha_j \le (1+Ch)\alpha_{j-1} + Ch^2\sum_{i=1}^{j-1}\alpha_i + Ch, \quad 2\le j\le l. \tag{11}$$

Then

$$\alpha_j \le (1+Ch)^j[\alpha_1 + jCh^2\sum_{i=1}^{j-1}\alpha_i + jCh], \quad 2\le j\le l.$$

Proof From (11)

$$\begin{aligned}\alpha_{j-1} &\le (1+Ch)\alpha_{j-2} + Ch^2\sum_{p=1}^{j-2}\alpha_p + Ch\\ &\le (1+Ch)\alpha_{j-2} + Ch^2\sum_{p=1}^{j-1}\alpha_p + Ch.\end{aligned} \tag{12}$$

Putting in (11)

$$\alpha_j \le (1+Ch)^2\alpha_{j-2} + Ch^2[1+(1+Ch)]\sum_{p=1}^{j-1}\alpha_p + Ch[1+(1+Ch)]. \tag{13}$$

By repeating the above process we get

$$\begin{aligned}\alpha_j \le\ & (1+Ch)^{(j-1)}\alpha_1 + Ch^2[1+(1+Ch)+\cdots+(1+Ch)^{(j-1)}]\sum_{p=1}^{j-1}\alpha_p\\ & +Ch[1+(1+Ch)+\cdots+(1+Ch)^{(j-1)}]\\ \le\ & (1+Ch)^j[\alpha_1 + jCh^2\sum_{p=1}^{j-1}\alpha_p + jCh].\end{aligned} \tag{14}$$

This completes the proof of the lemma. □

Lemma 4.4 *There exists a constant C independent of j, h and l such that*

$$|||\delta u_j^l||| \le C.$$

Proof From (8) for $j=1$ we get

$$\delta u_1^l - hA\delta u_1^l = Au_0 + F_1^l + hk_{10}^l Bu_0.$$

Lemma 4.1 implies that

$$|||\delta u_1^l||| \le |||Au_0 + F_1^l + hk_{10}^l Bu_0||| \le C.$$

Hence $|||Au_1^l|||\le C$. Let $2\le j\le l$. Subtracting (8) for $j-1$ from (8) for j, we get

$$\delta u_j^l - hA\delta u_j^l = \delta u_{j-1}^l + F_j^l - F_{j-1}^l + hk_{jj-1}^l Bu_{j-1}^l + \sum_{i=0}^{j-2}[k_{ji}^l - k_{j-1i}^l]Bu_i^l.$$

Applying Lemma 4.1 again, we get

$$\begin{aligned}|||\delta u_j^l||| &\leq |||\delta u_{j-1}^l||| + |||F_j^l - F_{j-1}^l||| + h|k_{j(j-1)}^l||||Bu_{j-1}^l||| \\ &\quad + h\sum_{i=0}^{j-2}|k_{ji}^l - k_{(j-1)i}^l||||Bu_i^l||| \\ &\leq (1+Ch)|||\delta u_{j-1}^l||| + Ch^2\sum_{i=0}^{j-2}|||\delta u_i^l||| + Ch^2\sum_{i=0}^{j-1}|||Au_i^l||| + Ch \\ &\leq (1+Ch)\max_{1\leq p\leq j-1}|||\delta u_p^l||| + Ch^2\sum_{i=0}^{j-1}|||Au_i^l||| + Ch \qquad (15)\end{aligned}$$

From (8), for $2 \leq i \leq j$, we have

$$\begin{aligned}|||Au_i^l||| &\leq |||\delta u_i^l||| + |||F_i^l||| + Ch\sum_{p=1}^{i-1}|||Bu_p^l||| \\ &\leq C(1 + \max_{1\leq p\leq i}|||\delta u_p^l|||) + Ch + Ch\sum_{p=1}^{i-1}|||Au_p^l|||. \qquad (16)\end{aligned}$$

Applying Lemma 4.2 in (16), we get

$$|||Au_i^l||| \leq Ce^{CT}(1 + \max_{1\leq p\leq i}|||\delta u_p^l|||). \qquad (17)$$

Using (17) in (15), we have

$$\begin{aligned}\max_{1\leq p\leq j}|||\delta u_p^l||| &\leq (1+Ch)\max_{1\leq p\leq j-1}|||\delta u_p^l||| \\ &\quad + Ch^2\sum_{p=1}^{j-1}\max_{1\leq p\leq i}|||\delta u_p^l||| + Ch. \qquad (18)\end{aligned}$$

To use Lemma 4.3 in (18), we take $\alpha_j = \max_{1\leq p\leq j}|||\delta u_p^l|||$ and the fact that $(1+Ch)^j \leq e^{CT}$, $2 \leq j \leq l$ and $\alpha_1 \leq C$ to get the estimate

$$\max_{1\leq p\leq j}|||\delta u_p^l||| \leq C + Ch\sum_{p=1}^{j-1}\max_{1\leq p\leq j-1}|||\delta u_p^l|||. \qquad (19)$$

Again we apply Lemma 4.2 to get the required estimate. This completes the proof of the lemma. □

Now, using the discrete points u_j^l, we introduce the following sequences of functions defined from $[0,T]$ into H.

Definition 4.1 We define the Rothe sequence $\{U^l\} \subset C([0,T];H)$ given by

$$U^l(t) = u_{j-1}^l + (t - t_{j-1}^l)\delta u_j^l, \quad t \in [t_{j-1}^l, t_j^l],\ j = 1,2,\ldots,l.$$

Definition 4.2 We define the sequence $\{X^l\}$ of step function from $[0,T]$ into H given by

$$X^l(0) = u_0^l, \quad X^l(t) = u_{j-1}^l, \quad t \in (t_{j-1}^l, t_j^l],\ j = 1,2,\ldots,n,\ l.$$

Remark 4.2 Each of the functions $\{U^l\}$ is Lipschitz continuous with uniform Lipschitz constant, i.e.,

$$|||U^l(t) - U^l(s)||| \leq C|t-s|, \quad t, s \in [0,T].$$

Furthermore,

$$|||U^l(t) - X^l(t)||| \leq \frac{C}{l}.$$

From (8), for $t \in (0,T]$, we may write

$$\frac{d^-}{dt}U^l(t) - AX^l(t) = f^l(t) + \int_0^t K^l(s)ds \tag{20}$$

where $f^l(t) = F(t_j^l, X^l(t))$ for $t \in [t_{j-1}^l, t_j^l]$ and

$$K^l(0) = 0, \quad K^l(t) = h\sum_{i=0}^{j-1} k_{ji}^l B u_i^l, \quad t \in (t_{j-1}^l, t_j^l]. \tag{21}$$

In the next lemma we prove the local uniform convergence of the Rothe sequence.

Lemma 4.5 *There exist a subsequence $\{U^{l_k}\}$ of $\{U^l\}$ and a function $u : [0,T] \to D(A)$ such that $U^{l_k} \to u$ in $C([0,T];H)$, and $AU^{l_k}(t) \rightharpoonup Au(t)$ uniformly in H as $k \to \infty$ where $\rightharpoonup$ denotes the weak convergence in H. Furthermore, $Au(t)$ is weakly continuous on $[0,T]$.*

Proof Since $\{U^l(t)\}$ and $\{AX^l(t)\}$ are uniformly bounded in the Hilbert space H, there exist weakly convergent subsequences $\{U^{l_k}(t)\}$ and $\{AX^{l_k}(t)\}$ (we take the same indices without loss of generality otherwise we first take the subsequence $\{U^{l_k}(t)\}$ of $\{U^l(t)\}$ and then take the subsequence $\{U^{l_{k_l}}(t)\}$ and $\{AX^{l_{k_l}}(t)\}$ of $\{U^{l_k}(t)\}$ and $\{AX^{l_k}(t)\}$, respectively). Thus, there exist functions $u, w : [0,T] \to H$ such that $U^{l_k}(t) \rightharpoonup u(t)$ and $AX^{l_k}(t) \rightharpoonup w(t)$ as $k \to \infty$. Also, we have $X^{l_k}(t) \rightharpoonup u(t)$ as $k \to \infty$. Let χ_Q be the characteristic function of a set $Q \subset \mathbb{R}^n$ and let $B_r = \{x \in \mathbb{R}^n : |x| < r\}$, $r > 0$, be the open ball in $\mathbb{R}^n$ of radius r centered at the origin.

Let

$$\begin{aligned} X_r^l(t) &= (\chi_{B_r}X_1^l(t), \chi_{B_r}X_2^l(t)), \quad l = 1, 2, \ldots, \\ u_r(t) &= (\chi_{B_r}u_1(t), \chi_{B_r}u_2(t)), \\ w_r(t) &= (\chi_{B_r}w_1(t), \chi_{B_r}w_2(t)). \end{aligned}$$

Clearly, $\{X_r^{l_k}(t)\}$ and $\{AX_r^{l_k}(t)\}$ are uniformly bounded and $X_r^{l_k}(t) \rightharpoonup u_r(t)$ and $AX_r^{l_k}(t) \rightharpoonup w_r(t)$ as $k \to \infty$. Since $\chi_{B_r}X_1^{l_k}(t) \in H^2(B_{r+\epsilon}) \cap H_0^1(B_{r+\epsilon})$, by the equivalence of the norms $\|u\|_{H^2(B_{r+\epsilon})\cap H_0^1(B_{r+\epsilon})}$ and $\|\Delta u\|_{L^2(B_{r+\epsilon})}$ in $H^2(B_{r+\epsilon}) \cap H_0^1(B_{r+\epsilon})$ (cf. inequality (6.3.9) on page 214 in Atkinson and Han [1]), it follows that $(\partial_{x_i}\chi_{B_r}X_1^l(t), \chi_{B_r}X_2^l(t)) \in D^1(\mathbb{R}^n) \times D^1(\mathbb{R}^n)$, for $i = 1, 2, \ldots, n$, where ∂_{x_i} is the distributional partial derivative with respect to the variable x_i and $D^1(\mathbb{R}^n)$ is the space introduced by Lieb and Loss [11] in the sense that a function $f \in D^1(\mathbb{R}^n)$ if it is in $L^1_{loc}(\mathbb{R}^n)$, its distributional derivative $\partial_{x_i}f$ is in $L^2(\mathbb{R}^n)$, for $i = 1, 2, \ldots, n$, and f vanishes at infinity. Hence we may apply Theorem 8.6 of Lieb and Loss [11] to conclude that $X_r^{l_k}(t) \to u_r(t)$ as $k \to \infty$ and hence $U_r^{l_k}(t) \to u_r(t)$ as $k \to \infty$ in H.

Now, For $\epsilon > 0$, there exist $r_\epsilon > 0$ such that

$$|||U^{l_k}(t) - u(t)|||^2 < |||U_r^{l_k}(t) - u_r(t)|||^2 + \frac{\epsilon}{2}.$$

Now we choose k_0 sufficiently large such that

$$|||U_r^{l_k}(t) - u_r(t)|||| < \frac{\epsilon}{2}, \quad k \geq k_0.$$

Hence

$$|||U^{l_k}(t) - u_r(t)||| < \epsilon, \quad k \geq k_0.$$

Thus $U^{l_k}(t) \to u(t)$ as $k \to \infty$ in H. Since U^{l_k} is Lipschitz continuous with uniform Lipschitz constant, it follows that $\{U^{l_k}\}$ is equi-continuous in $C([0,T];H)$ and $\{U^{l_k}(t)\}$ is relatively compact in H. Hence by Asoli-Arzela theorem, $U^{l_k} \to u$ as $k \to \infty$ in $C([0,T];H)$. From the properties of the operator A we have $u(t) \in D(A)$ and $Au(t) = w(t)$. To show the weak continuity of $Au(t)$ in t, let $\{t_k\} \subset [0,T]$ such that $t_k \to t$ as $k \to \infty$, $t \in [0,T]$. Then $u(t_k) \to u(t)$ and since $|||Au(t_k)||| \leq C$, there exists a subsequence $\{Au(t_{k_p})\} \subset \{Au(t_k)\}$ such that $Au(t_{k_p}) \rightharpoonup z(t)$ as $p \to \infty$. Since $u(t_{k_p}) \to u(t)$ and $Au(t_{k_p}) \rightharpoonup z(t)$ as $p \to \infty$, it follows as above that $u(t) \in D(A)$ and $Au(t) = z(t)$. Hence $Au(t)$ is weakly continuous. This completes the proof of the lemma. □

Remark 4.3 Since $Bx = Ax + Px$ for $x \in D(A)$ and $P : H^1 \times H^0 \to H$ is a bounded linear operator, $Bu(t)$ is weakly continuous on $[0,T]$.

Lemma 4.6 *$Au(t)$ and $Bu(t)$ are Bochner integrable on $[0,T]$.*

For the proof of this lemma we refer to Bahuguna and Raghavendra [4]

Lemma 4.7 *Let $\{K^l(t)\}$ be the sequence of functions defined by (21) and*

$$K(\phi)(t) = \int_0^t k(t-s)\phi(s)ds,$$

where $\phi : [0,T] \to H$ is Bochner integrable. We have

$$K^{l_k}(t) \rightharpoonup K(Bu)(t),$$

uniformly on $[0,T]$ as $k \to \infty$.

Proof We first show that $K^{l_k}(t) - K(X^{l_k})(t) \to 0$ uniformly on $[0,T]$ as $p \to \infty$. For $t \in (t_{j-1}^{l_k}, t_j^{l_k}]$, we have

$$\begin{aligned} K^{l_k}(t) - K(X^{l_k})(t) &= h\sum_{i=0}^{j-1} k_{ji}^{l_k} Bu_i^{l_k} - \int_0^t k(t-s)BX^{l_k}(s)\, ds \\ &= \sum_{i=0}^{j-2}\int_{t_i^{l_k}}^{t_{i+1}^{l_k}} [k_{ji}^{l_k} - k(t-s)\, ds]Bu_{i+1}^{l_k} \\ &\quad -[\int_{t_{j-1}^{l_k}}^{t} k(t-s)\, ds]Bu_j^{l_k}. \end{aligned}$$

Since $|||Bu_j^{l_k}||| \leq C$, we have

$$\begin{aligned} |||K^{l_k}(t) - K(X^{l_k})(t)||| &\leq C\sum_{i=0}^{j-2}\int_{t_i^{l_k}}^{t_{i+1}^{l_k}} |k_{ji}^{l_k} - k(t-s)|ds \\ &\leq C\int_{t_{j-1}^{l_k}}^{t} |k(t-s)|ds \to 0, \quad \text{as } k \to \infty. \end{aligned}$$

Now we show that $K(X^{l_k})(t) \rightharpoonup \int_0^t k(t-s)Bu(s)\, ds$ uniformly as $k \to \infty$. For any $v \in H$, We note that $(Bu(t), v)$ is continuous hence we may write

$$\left(\int_0^t k(t-s)Bu(s)\, ds, v\right) = \int_0^t k(t-s)(Bu(s), v)\, ds.$$

Now, for any $v \in H$,

$$\begin{aligned} (K(X^{l_k})(t), v) &= \left(\int_0^t k(t-s)BX^{l_k}(s)ds, v\right) \\ &= \sum_{i=0}^{j-2}\int_{t_i^{l_k}}^{t_{i+1}^{l_k}} k(t-s)(BX^{l_k}(s), v)ds \\ &\quad \int_{t_{j-1}^{l_k}}^{t} k(t-s)(BX^{l_k}(s), v)ds \to \int_0^t k(t-s)(Bu(s), v)\, ds, \end{aligned}$$

as $k \to \infty$. This completes the proof of the lemma. □

5 Proof of Theorem 3.1

In this section we prove Theorem 3.1 with the help of lemmas of the previous section.

Proof Let $v \in H$ be any element. For $t \in (0, T]$, we have

$$(U^{l_k}(t), v) - (Au(t), v) = (u_0, v) + \int_0^t (K^{l_k}(s) + f^{l_k}(s), v)\, ds.$$

Passing to the limit as $p \to \infty$ using bounded convergence theorem and Lemmas 4.5 and 4.7, we have

$$(u(t), v) - (Au(t), v) = (u_0, v) + \int_0^t (K(u)(s) + F(s, u(s)), v)\, ds.$$

Using the continuity of the integrands on the right hand side, we have $(u(t), v)$ is continuously differentiable and

$$\frac{d}{dt}(u(t), v) - (Au(t), v) = (F(t, u(t)) + \int_0^t (k(t-s)Bu(s)\, ds, v). \tag{22}$$

Since $u(t)$ is differentiable a.e. on $[0, T]$, we may take $\frac{d}{dt}$ inside the inner product for a.e. $t \in [0, T]$, hence

$$\frac{du(t)}{dt} - Au(t) = F(t, u(t)) + \int_0^t k(t-s)Bu(s)\, ds, \quad \text{a.e. } t \in [0, T],\ u(0) = u_0. \tag{23}$$

Now we prove the uniqueness under the assumption that $k \in C^1[0,T]$. Let u_1 and u_2 be two solutions of (7) and let $u = u_1 - u_2$. Then

$$\begin{aligned} u(t) &= \int_0^t T(t-s)[F(s,u_1(s)) - F(s,u_2(s)) \\ &\quad + \int_0^s k(s-\tau)Au(\tau)\,d\tau]\,ds \\ &= \int_0^t T(t-s)[F(s,u_1(s)) - F(s,u_2(s))]ds \\ &\quad + \int_0^t \left(\int_0^s k(s-\tau)T(t-s)Au(\tau))\,d\tau\right) ds \\ &= \int_0^t T(t-s)[F(s,u_1(s)) - F(s,u_2(s))]ds \\ &\quad + \int_0^t \left(\int_\tau^t k(s-\tau)T(t-s)Au(\tau))\,ds\right) d\tau \\ &= \int_0^t T(t-s)[F(s,u_1(s)) - F(s,u_2(s))]ds \\ &\quad + \int_0^t \left(\int_0^{t-\tau} k(t-\eta-\tau)T(\eta)Au(\tau))\,d\eta\right) d\tau. \end{aligned} \tag{24}$$

Since $u(\tau) \in D(A)$ for $\tau \in [0,T]$, we have $T(\eta)Au(\tau) = \frac{\partial}{\partial\eta}(T(\eta)u(\tau))$ (cf. Theorem 1.2.4 in Pazy). Thus, we have

$$\begin{aligned} u(t) &= \int_0^t T(t-s)[F(s,u_1(s)) - F(s,u_2(s))]ds \\ &\quad + \int_0^t \left(\int_0^{t-\tau} k(t-\eta-\tau)\frac{\partial}{\partial\eta}(T(\eta)u(\tau))\,d\eta\right) d\tau \\ &= \int_0^t T(t-s)[F(s,u_1(s)) - F(s,u_2(s))]ds \\ &\quad + k(0)\int_0^t T(t-\tau)u(\tau)d\tau - \int_0^t k(t-\tau)u(\tau)d\tau \\ &\quad + \int_0^t \left(\int_0^{t-\tau} k'(t-\eta-\tau)T(\eta)u(\tau)d\eta\right) d\tau. \end{aligned} \tag{25}$$

Now taking the norm and using the fact that $\|T(t)\| \le 1$, we have

$$\max_{0\le r\le t} |||u(r)||| \le C\int_0^t \max_{0\le r\le s} |||u(r)|||\,ds.$$

Gronwall's inequality implies that $u(t) \equiv 0$. This completes the proof of the theorem. □

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Approximate Constraint-Following of Mechanical Systems under Uncertainty

Ye-Hwa Chen *

The George W. Woodruff School of Mechanical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332-0405, USA

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Abstract: We consider a mechanical system, which is required to obey a set of constraints. The system may contain uncertainty, which is possibly fast time-varying. We propose a robust control scheme that is motivated by the Nature's strategy. The control also takes into account the uncertainty for guaranteeing approximate constraint following.

Keywords: *Mechanical system; constraint; motion control; robust control.*

Mathematics Subject Classification (2000): 70Q05, 70E60, 93B52.

1 Introduction

For a mechanical system to be confined to a set of constraints, constraint forces are needed. Out of many possible forms for such forces, in Lagrangean mechanics, it is postulated that the constraint forces should be governed by the Lagrange's form of d'Alembert's principle. In a sense this is what Joseph-Louis Lagrange asserted the *Nature* would do ([1]).

In the past, the majority of the efforts in constrained mechanical systems can be divided into two categories: the *passive constraint problem* and the *servo constraint problem*. In the passive constraint problem, the main focus is to investigate what the *Nature* will do in order to assure that the constraints are (strictly) obeyed. These include, for example, the *Maggi equation* [2,3], the *Boltzmann and Hamel equation* [3, 4], the *Gibbs*

* Corresponding author: yehwa.chen@me.gatech.edu

and Appell equation [4–6], and the *Udwadia and Kalaba equation* [7]. A more complete list can be found in [1].

In the servo constraint problem, on the other hand, the main focus is to find what the *engineer* should do, so that the constraints are followed. A survey on the use of geometric and algebraic approaches can be found in [8–12]. Most of the emphasis is on precise model-based control design. Prominent contributions which deal with uncertainty can be found in, e.g., [13–16] and their bibliographies.

In these two categories, the main differences are twofold. First, the *Nature* (i.e., the environment or the structure) is assumed to possess all information, be it the system or the surrounding. Therefore there is no uncertainty. The engineer, however, is always limited to a rather confined domain of knowledge. As a result, uncertainty tends to be inevitable in many applications.

Second, the *Nature* is only contended with a strict performance. That is, the constraints are to be precisely followed. The engineer, taking a more pragmatic viewpoint, can be settled with approximate constraint following.

This paper falls into the second category. The features of the current approach are fourfold. First, no state transformation is needed. One can choose any coordinate system to represent the system and design the control. Second, the uncertainty considered can be (possibly fast) time-varying. Third, no Lagrange multiplier is needed for control formulation; hence no force feedback. Fourth, no initial condition restrictions are imposed. The starting configuration of the mechanical system can be far away from the desired constraint.

2 Mechanical System Subject to Constraints

Consider the following mechanical system:

$$M(q(t),\sigma(t),t)\ddot{q}(t)+C(q(t),\dot{q}(t),\sigma(t),t)\dot{q}(t)+g(q(t),\sigma(t),t)=\tau(t). \tag{2.1}$$

Here $t\in R$ is the independent variable, $q\in R^n$ is the coordinate, $\dot{q}\in R^n$ is the velocity, $\ddot{q}\in R^n$ is the acceleration, $\sigma\in\Sigma\subset R^p$ is the uncertain parameter, and $\tau\in R^n$ is the control input. Furthermore, $M(q,\sigma,t)$ is the inertia matrix, $C(q,\dot{q},\sigma,t)\dot{q}$ is the Coriolis/centrifugal force, and $g(q,\sigma(t),t)$ is the gravitational force. The matrices/vector $M(q,\sigma,t)$, $C(q,\dot{q},\sigma,t)$, and $g(q,\sigma,t)$ are of appropriate dimensions. We assume that the functions $M(\cdot)$, $C(\cdot)$, and $g(\cdot)$ are continuous (this can be generalized to be Lebesgue measurable in t). In addition, the bounding set Σ is prescribed and compact.

Remark 2.1 The coordinate q can be selected based on the specifics of the problem and does not need to be the *generalized coordinate* [17].

The following constraints are proposed:

$$\sum_{i=1}^{n}A_{li}(q,t)\dot{q}_i=c_l(q,t),\quad l=1,\ldots,m, \tag{2.2}$$

where $\dot{q}_i$ is the i-th component of $\dot{q}$, $A_{li}(\cdot)$ and $c_l(\cdot)$ are both C^1, $m\le n$. They are the *first order* form of the constraints. The constraints may not be integrable; and may be nonholonomic in general. The constraints can be put in matrix form

$$A(q,t)\dot{q}=c(q,t), \tag{2.3}$$

where $A = [A_{li}]_{m\times n},\ c = [c_1, c_2, \ldots, c_m]^{\mathrm{T}}$.

We now convert the first order form into *second order* form. Differentiating the constraint equations (2.2) with respect to t yields

$$\sum_{i=1}^{n}\left(\frac{d}{dt}A_{li}(q,t)\right)\dot{q}_i + \sum_{i=1}^{n} A_{li}(q,t)\ddot{q}_i = \frac{d}{dt}c_l(q,t), \tag{2.4}$$

where

$$\frac{d}{dt}\,A_{li}(q,t) = \sum_{k=1}^{n}\frac{\partial A_{li}(q,t)}{\partial q_k}\dot{q}_k + \frac{\partial A_{li}(q,t)}{\partial t},$$
$$\frac{d}{dt}\,c_l(q,t) = \sum_{k=1}^{n}\frac{\partial c_l(q,t)}{\partial q_k}\dot{q}_k + \frac{\partial c_l(q,t)}{\partial t}.$$

Equation (2.4), the second order form of the constraints can be rewritten as

$$\sum_{i=1}^{n} A_{li}(q,t)\ddot{q}_i = -\sum_{i=1}^{n}\left(\frac{d}{dt}A_{li}(q,t)\right)\dot{q}_i + \frac{d}{dt}c_l(q,t) = b_l(q,\dot{q},t), \quad l = 1,\ldots,m, \tag{2.5}$$

or in matrix form

$$A(q,t)\ddot{q} = b(q,\dot{q},t), \tag{2.6}$$

where $b = [b_1, b_2, \ldots, b_m]^{\mathrm{T}}$.

Remark 2.2 For a given configuration, the *possible* velocity $\dot{q}$ is governed by (2.3) while the *possible* acceleration $\ddot{q}$ is governed by (2.6) ([17]). Advantages of considering second order constraints are discussed in, e.g., [18,19].

Remark 2.3 The constraint (2.6) is in fact a very general form. It includes typical constraints as illustrated in, e.g., [1,17]. It also includes a number of standard control objectives such as stability, trajectory tracking, and optimality. All one needs is to prescribe a desirable system dynamics and then convert it to the second order form.

Remark 2.4 Besides the "practical" constraint a dynamic system needs to meet, which are thoroughly discussed in, e.g., [1], there is also the "numerical" constraint. By this, we mean that it is possible that a system, under a prescribed constraint, while in numerical simulations, tends to have a numerical drift of constraints and integrals. Therefore it is desirable to impose an additional "numerical" constraint on the system to make sure there is no numerical drift. The standard technique such as in [20] can be applied. The "numerical" constraint can be combined with the "practical" constraint to form (2.6).

3 The Passive Constraint Problem

There are two categories of problems associated with constraints. In the *passive constraint problem*, the environment (or the structure) is to supply the constraint force in order for the system to comply with the constraint. In the *active (or servo) constraint problem*, the control input supplies the required force. We discuss the first in this section.

The *virtual displacement* [17] δq is such that $A(q,t)\delta q = 0$. Let $Q^c \in R^n$ denote the constraint force. The *Lagrange's form of d'Alembert's principle*, which in turn prescribes the constraint to be *ideal*, is such that the first order constraint virtual work vanishes:

$$\delta' W^{(c)} = Q^{c\mathrm{T}} \delta q = 0, \tag{3.1}$$

where $\delta' W^{(c)}$ is the (first order) constraint virtual work.

Assumption 1 For each $(q,t) \in R^n \times R$, $\sigma \in \Sigma$, $M(q,\sigma,t) > 0$.

Remark 3.1 The assumption on the positive definiteness of the inertia matrix will be vital in later development. In the past, it was often believed that this was always true, and therefore a fact rather than an assumption. However, there are counter examples, as listed in [21], when q is not selected to be the generalized coordinate (as the current case).

Definition 3.1 For given A and b, the constraint (2.6) is called *consistent* if there exists at least one solution $\ddot{q}$.

Assumption 2 The constraint (2.6) is consistent.

Theorem 3.1 ([7, p. 233]) *Consider the system (2.1) and the constraint (2.6). Subject to Assumptions 3.1 and 3.2, the constraint force*

$$\begin{aligned} Q^c = & M^{1/2}(q,\sigma,t)(A(q,t)M^{-1/2}(q,\sigma,t))^{+} \\ & \times [b(q,\dot{q},t) + A(q,t)M^{-1}(q,\sigma,t)(C(q,\dot{q},\sigma,t)\dot{q} + g(q,\sigma,t))]. \end{aligned} \tag{3.2}$$

obeys the Lagrange's form of d'Alembert's principle (3.1) and renders the system to meet the constraint. Here "+" stands for the Moore-Penrose generalized inverse ([22, p. 337]).

Sketch of Proof: By the choice of (3.2), it can be shown that, with $\tau = Q^c$ in (2.1),

$$A\ddot{q} - b = A[M^{-1}(-C\dot{q} - g) + M^{-1}Q^c] - b = 0. \tag{3.3}$$

Furthermore, we have $Q^c \in \mathcal{R}(A^{\mathrm{T}})$ (note that $\delta q \in \mathcal{N}(A)$ and $\mathcal{R}(A^{\mathrm{T}}) \perp \mathcal{N}(A)$). □

Remark 3.2 The Lagrange's form of d'Alembert's principle renders the constraint force (3.2) to be the one with *minimum norm*, out of all possible alternative forces which can also meet (2.6) [7].

Remark 3.3 Theorem 1 suggests the strategy the *Nature* will undertake to meet the constraint. The constraint force is *model-based*. That is, it is based on the exact model information. Based on the theorem, one could apply the control input $\tau = Q^c$ to drive the system to meet (2.6), if the uncertainty *was* known. A more realistic consideration that the uncertainty is unknown is investigated in the next section.

4 Robust Servo Control Design

We now take the uncertainty into account while designing the control τ. Decompose the M, C, and g as follows:

$$\begin{aligned} M(q,\sigma,t) &= \bar{M}(q,t) + \Delta M(q,\sigma,t), \\ C(q,\dot{q},\sigma,t) &= \bar{C}(q,\dot{q},t) + \Delta C(q,\dot{q},\sigma,t), \\ g(q,\sigma,t) &= \bar{g}(q,t) + \Delta g(q,\sigma,t). \end{aligned} \tag{4.1}$$

Here $\bar{M}$, $\bar{C}$, and $\bar{g}$ denote the "nominal" portions with $\bar{M} > 0$ (this is always feasible since it is the designer's discretion), while ΔM, ΔC and Δg are the uncertain portions. The functions $\bar{M}(\cdot)$, $\Delta M(\cdot)$, $\bar{C}(\cdot)$, $\Delta C(\cdot)$, $\bar{g}(\cdot)$, and $\Delta g(\cdot)$ are all continuous. Let $D(q,t) := \bar{M}^{-1}(q,t)$, $\Delta D(q,\sigma,t) := M^{-1}(q,\sigma,t) - \bar{M}^{-1}(q,t)$, $E(q,\sigma,t) := \bar{M}(q,t)M^{-1}(q,\sigma,t) - I$ (hence $\Delta D(q,\sigma,t) = D(q,t)E(q,\sigma,t)$).

Assumption 3 For each $(q,t) \in R^n \times R$, $A(q,t)$ is of full rank.

Assumption 4 There exists $\hat{\rho}_E(\cdot)$: $R^n \times R \to (-1,\infty)$ such that for all $(q,t) \in R^n \times R$,

$$\frac{1}{2} \min_{\sigma\in\Sigma} \lambda_m \left(E(q,\sigma,t) + E^{\mathrm{T}}(q,\sigma,t)\right) \geq \hat{\rho}_E(q,t). \tag{4.2}$$

Remark 4.1 Suppose there is no uncertainty in M: $\bar{M} = M$, then $E = 0$ and hence one can choose $\hat{\rho}_E = 0$ to meet the assumption. By continuity, there is a (unidirectional) threshold for the allowable uncertainty in E. We note that a standard assumption in this area (see, e.g., [23]) that $\max_{\sigma\in\Sigma} \|E(q,\sigma,t)\| < 1$ for all $(q,t) \in R^n \times R$ is more restrictive than the current setting.

Assumption 5 For given $P \in R^{n\times n}$, $P > 0$, let

$$\Psi(q,t) := PA(q,t)D(q,t)D(q,t)A^{\mathrm{T}}(q,t)P.$$

There exists a scalar constant $\underline{\lambda} > 0$ such that

$$\inf_{(q,t)\in R^n\times R} \lambda_m\left(\Psi(q,t)\right) \geq \underline{\lambda}. \tag{4.3}$$

Remark 4.2 Under Assumptions 3.1, 3.2, and 4.1, the matrix $\Psi(q,t)$ is always positive definite. Thus all this assumption adds is that $\lambda_m(\Psi)$ is positively bounded from below.

Remark 4.3 Let $\beta(q,\dot{q},t) := A(q,t)\dot{q} - c(q,t)$. We consider the approximate constraint following problem. That is, it is possible that $\beta \neq 0$ (hence $A\ddot{q} \neq b$). This may be due to modelling uncertainty (and hence (3.2) can not be implemented by the designer; while it can be by the *Nature*). In addition, the system may not start with the constraint manifold in the beginning (i.e., $\beta \neq 0$ as $t = t_0$).

Consider the following control design:

$$\tau(t) = p_1(q(t),\dot{q}(t),t) + \hat{p}_2(q(t),\dot{q}(t),t) + \hat{p}_3(q(t),\dot{q}(t),t), \tag{4.4}$$

with

$$\hat{p}_2(q,\dot{q},t) = -\kappa \bar{M}^{-1}(q,t)A^{\mathrm{T}}(q,t)P(A(q,t)\dot{q} - c(q,t)), \tag{5}$$
$$\hat{p}_3(q,\dot{q},t) = -\hat{\gamma}(q,\dot{q},t)\hat{\mu}(q,\dot{q},t)\hat{\rho}(q,\dot{q},t), \tag{6}$$

where $\epsilon, \kappa \in R,\ \epsilon, \kappa > 0$,

$$\hat{\gamma}(q,\dot{q},t) = \begin{cases} \dfrac{(1+\hat{\rho}(q,t))^{-1}}{\|\hat{\mu}(q,\dot{q},t)\|}, & \text{if} \quad \|\hat{\mu}(q,\dot{q},t)\| > \epsilon, \\ \dfrac{(1+\hat{\rho}(q,t))^{-1}}{\epsilon}, & \text{if} \quad \|\hat{\mu}(q,\dot{q},t)\| \leq \epsilon, \end{cases} \tag{7}$$

$$\hat{\mu}(q,\dot{q},t) := \bar{M}^{-1}(q,t)A^{\mathrm{T}}(q,t)P(A(q,t)\dot{q} - c(q,t))\hat{\rho}(q,\dot{q},t). \tag{8}$$

The function $\hat{\rho}(\cdot) : R^n \times R^n \times R \to R_+$ is chosen such that for all $(q,\dot{q},t) \in R^n \times R^n \times R$,

$$\begin{aligned}\hat{\rho}(q,\dot{q},t) \geq \max_{\sigma\in\Sigma} \| &PA(q,t)\Delta D(q,\sigma,t)(-C(q,\dot{q},t)\dot{q} - g(q,t) + p_1(q,\dot{q},t) + \hat{p}_2(q,\dot{q},t)) \\ &- PA(q,t)D(q,t)(\Delta C(q,\dot{q},\sigma,t)\dot{q} + \Delta g(q,\sigma,t))\|.\end{aligned} \tag{4.9}$$

Theorem 4.1 *Subject to Assumptions 3.1, 3.2, and 4.1-4.3, consider the system (2.1). The control (4.4) renders* β *uniformly bounded (that is, for any* $r > 0$*, there is a* $d(r) < \infty$ *such that if* $\|\beta(q(t_0),\dot{q}(t_0),t_0)\| \leq r$*, then* $\|\beta(q(t),\dot{q}(t),t)\| \leq d(r)$ *for all* $t \geq t_0$*) and uniformly ultimately bounded (that is, for any* $r > 0$ *and* $\underline{d} > 0$ *with* $\|\beta(q(t_0),\dot{q}(t_0),t_0)\| \leq r$*,* $\|\beta(q(t),\dot{q}(t),t)\| \leq \bar{d}$ *for any* $\bar{d} > \underline{d}$ *and all* $t \geq t_0 + T(\bar{d},r)$*, where* $T(\bar{d},r) < \infty$*). Furthermore,* $\bar{d} \to 0$ *as* $\epsilon \to 0$*.*

Proof Let $V(\beta) = \beta^{\mathrm{T}}P\beta$. For any given $\sigma(\cdot)$, the derivative of V along a trajectory is evaluated as (for simplicity, arguments of functions are sometimes omitted when no confusions are likely to arise):

$$\dot{V} = 2\beta^{\mathrm{T}}P(A\ddot{q} - b) = 2\beta^{\mathrm{T}}P\left\{A\left[M^{-1}(-C\dot{q} - g) + M^{-1}(p_1 + \hat{p}_2 + \hat{p}_3)\right] - b\right\}. \tag{4.10}$$

After decomposing M^{-1}, C, and g, we have

$$\begin{aligned} &A[M^{-1}(-C\dot{q} - g) + M^{-1}(p_1 + \hat{p}_2 + \hat{p}_3)] - b \\ &\quad = A[(D+\Delta D)(-\bar{C}\dot{q} - \bar{g} - \Delta C\dot{q} - \Delta g) + (D+\Delta D)(p_1 + \hat{p}_2 + \hat{p}_3)] - b \\ &\quad = A[D(-\bar{C}\dot{q} - \bar{g} + p_1 + \hat{p}_2) + D(-\Delta C\dot{q} - \Delta g) + \Delta D(-C\dot{q} - g + p_1 + \hat{p}_2) \\ &\qquad + (D+\Delta D)\hat{p}_3] - b. \end{aligned}$$

First, we recall that

$$A[D(-\bar{C}\dot{q} - \bar{g}) + Dp_1] - b = 0. \tag{4.11}$$

Next, by (4.9),

$$\begin{aligned} &\beta^{\mathrm{T}}PA[D(-\Delta C\dot{q} - \Delta g) + \Delta D(-C\dot{q} - g + p_1 + \hat{p}_2)] \\ &\qquad \leq 2\|\beta\|\|PA[D(-\Delta C\dot{q} - \Delta g) + \Delta D(-C\dot{q} - g + p_1 + \hat{p}_2)]\| \leq 2\|\beta\|\hat{\rho}. \end{aligned} \tag{4.12}$$

Based on (4.5),

$$2\beta^{\mathrm{T}}PAD\hat{p}_2 = 2\beta^{\mathrm{T}}PAD(-\kappa\bar{M}^{-1}A^{\mathrm{T}}P(A\dot{q} - c)) = -2\kappa\eta^{\mathrm{T}}\eta = -2\kappa\|\eta\|^2, \tag{4.13}$$

where $\eta = \bar{M}^{-1}A^{\mathrm{T}}P\beta$. By $\Delta D = DE$, (4.6), and recalling that $\bar{M}^{-1} = D$,

$$\begin{aligned} 2\beta^{\mathrm{T}}PA(D+\Delta D)\hat{p}_3 &= 2\beta^{\mathrm{T}}PA(D+DE)(-\hat{\gamma}\hat{\mu}\hat{\rho}) \\ &= 2(DA^{\mathrm{T}}P\beta\hat{\rho})^{\mathrm{T}}(I+E)(-\hat{\gamma}\hat{\mu}) = 2\hat{\mu}^{\mathrm{T}}(I+E)(-\hat{\gamma}\hat{\mu}) \\ &= -2\hat{\gamma}\hat{\mu}^{\mathrm{T}}\hat{\mu} - 2\hat{\gamma}\hat{\mu}^{\mathrm{T}}E\hat{\mu} \le -2\hat{\gamma}\|\hat{\mu}\|^2 - \hat{\gamma}\lambda_m(E+E^{\mathrm{T}})\|\hat{\mu}\|^2 \\ &\le -2\hat{\gamma}(1+\hat{\rho}_E)\|\hat{\mu}\|^2. \end{aligned} \tag{4.14}$$

As $\|\hat{\mu}\| > \epsilon$, by (4.7),

$$-2\hat{\gamma}(1+\hat{\rho}_E)\|\hat{\mu}\|^2 = -2\,\frac{(1+\hat{\rho}_E)^{-1}}{\|\hat{\mu}\|}\,(1+\hat{\rho}_E)\|\hat{\mu}\|^2 = -2\|\hat{\mu}\|. \tag{4.15}$$

As $\|\hat{\mu}\| \le \epsilon$, then by (4.7),

$$-2\hat{\gamma}(1+\hat{\rho}_E)\|\hat{\mu}\|^2 = -2\,\frac{(1+\hat{\rho}_E)^{-1}}{\epsilon}\,(1+\hat{\rho}_E)\|\hat{\mu}\|^2 = -2\,\frac{\|\hat{\mu}\|^2}{\epsilon}\,. \tag{4.16}$$

With (4.11)–(4.15), we have for $\|\hat{\mu}\| > \epsilon$,

$$\dot{V} \le -2\kappa\|\eta\|^2 - 2\|\hat{\mu}\| + 2\|\beta\|\hat{\rho} = -2\kappa\|\eta\|^2 - 2\|\hat{\mu}\| + 2\|\hat{\mu}\| = -2\kappa\|\eta\|^2.$$

As $\|\hat{\mu}\| \le \epsilon$,

$$\dot{V} \le -2\kappa\|\eta\|^2 - 2\,\frac{\|\hat{\mu}\|^2}{\epsilon} + 2\|\hat{\mu}\| \le -2\kappa\|\eta\|^2 + \frac{\epsilon}{2}\,.$$

Finally we conclude that

$$\dot{V} \le -2\kappa\|\eta\|^2 + \frac{\epsilon}{2}\,.$$

By Rayleigh's principle ([21]) and Assumption 4.3,

$$\|\eta\|^2 = \eta^{\mathrm{T}}\eta = \beta^{\mathrm{T}}PADDA^{\mathrm{T}}P\beta \ge \lambda_m(PADDA^{\mathrm{T}}P)\|\beta\|^2 \ge \underline{\lambda}\|\beta\|^2.$$

Therefore,

$$\dot{V} \le -2\kappa\underline{\lambda}\|\beta\|^2 + \frac{\epsilon}{2}\,.$$

Upon invoking arguments as in [23], we conclude uniform boundedness with

$$d(r) = \begin{cases} \sqrt{\dfrac{\lambda_M(P)}{\lambda_m(P)}}\,R, & \text{if} \quad r \le R, \\ \sqrt{\dfrac{\lambda_M(P)}{\lambda_m(P)}}\,r, & \text{if} \quad r > R, \end{cases}$$

$$R = \sqrt{\frac{\epsilon}{4\kappa\underline{\lambda}}}\,.$$

Uniform ultimate boundedness also follows with

$$\underline{d} = \sqrt{\frac{\lambda_M(P)}{\lambda_m(P)}}\,R,$$

$$T(\bar{d}, r) = \begin{cases} 0, & \text{if} \quad r \le \bar{d}\sqrt{\dfrac{\lambda_m(P)}{\lambda_M(P)}}\,, \\ \dfrac{\lambda_M(P)r^2 - (\lambda_m^2(P)/\lambda_M(P))\bar{d}^2}{2\kappa\underline{\lambda}\bar{d}^2(\lambda_m(P)/\lambda_M(P)) - (\epsilon/2)}\,, & \text{otherwise.} \end{cases} \tag{4.17}$$

Based on (4.17), the size of the uniform ultimate boundedness region $\bar{d} \to 0$ as $\epsilon \to 0$. □

Remark 4.4 With the uncertainty in presence and no restrictions on the initial condition, it is only reasonable to expect approximate constraint following, which is shown in the ultimate boundedness of β. In the special case when there is no uncertainty, i.e., $\Delta D \equiv 0$, $\Delta C \equiv 0$, and $\Delta g \equiv 0$, one may choose $\rho = 0$ and hence $\hat{p}_3 = 0$. This means $\tau = p_1 + \hat{p}_2$ and $\dot{V} \leq -2\kappa\underline{\lambda}\|\beta\|^2$. One therefore expects $\beta \to 0$ as $t \to \infty$. If, in addition, we choose $\hat{p}_2 = 0$, then $\dot{V} = 0$. This means if $\beta = 0$ initially (i.e., the constraint is met initially), then $\beta = 0$ for all $t \geq t_0$. This special case falls into Theorem 3.1, the perfect constraint following case.

5 Conclusions

We consider a mechanical system subject to a class of (possibly nonholonomic) constraints. The system contains uncertainty. The control design objective is to render the system to follow the constraint sufficiently close, even in the presence of uncertainty. Two robust control designs are proposed. They are motivated by a previous design which is based on the Lagrange's form of D'Alembert's principle; hence the Nature's action. The controls assure the uniform boundedness and uniform ultimate boundedness of the tracking error (denoted by β).

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Neutral Functional Equations with Causal Operators on a Semi-Axis

C. Corduneanu [1*] and M. Mahdavi [2]

[1] *The University of Texas at Arlington, Department of Mathematics*
411 S. Nedderman Drive, Arlington, Texas 76019, U.S.A.
[2] *Bowie State University, Department of Mathematics*
14000 Jericho Park Road, Bowie, Maryland 20715, U.S.A.

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Abstract: This paper is concerned with the global study of a certain class of functional differential equation involving causal (abstract Volterra) operators on a certain function space $\mathrm{E}(R_+, R^n)$. It is closely related to our previous joint papers, listed in the References, the difference being motivated by the fact that we consider new function spaces on the half-axis R_+. The approach in this paper is also somewhat different than in preceding papers, by C. Corduneanu, the results being also different. A dynamical interpretation is also indicated.

Keywords: *Neutral functional equations; causal operators; global existence.*

Mathematics Subject Classification (2000): 34K05, 34K25, 34K40.

1 Statement of the Problem

Let us consider the functional differential equation

$$\frac{d}{dt}\left[\frac{dx(t)}{dt} - (Lx)(t)\right] = (Vx)(t), \quad t \in R_+, \tag{1}$$

where $x \in R^n$, $n \geq 1$ is an integer, and L, V are causal operators acting on the function space $C(R_+, R^n)$, consisting of all continuous maps from R_+ into R^n, the topology/convergence being defined by the family of semi-norms $\{|x|_k : \ k \geq 1\}$, with

* Corresponding author: concord@uta.edu

$|x_k| = \sup\{|x(t)|:\ 0 \le t \le k\}$, $k \ge 1$. By $|\cdot|$ we denote the Euclidean norm of the space R^n. As it was mentioned, our approach here is somewhat different than in [4, 5, 6].

It is well known (see, for instance [2, 3]) that the above topology/convergence means uniform convergence on any bounded interval $[0, T] \subseteq R_+$.

An initial condition of the form

$$x(0) = x^0 \in R^n, \quad \dot{x}(0) = v^0 \in R^n, \tag{2}$$

must be associated with (1), if we expect to get a unique solution to the Cauchy problem (1), (2).

Unlike our preceding papers, [8, 9, 12, 13], we will be interested here in finding global solutions to the problem (1), (2), i.e. belonging to the space $C(R_+, R^n)$.

The space $C(R_+, R^n)$ contains as subspaces (closed or not) many usual spaces appearing in the theory of differential or integral equations. An example is the space $BC(R_+, R^n)$ consisting of all bounded continuous maps from R_+ into R^n with the norm

$$\|x\| = \sup\{|x(t)|:\ t \in R_+\}, \tag{3}$$

$BC(R_+, R^n)$ is a Banach space. Other function spaces are encountered in the literature, such as the space $C_\ell(R_+, R^n)$, consisting of all maps in $BC(R_+, R^n)$ such that $\lim x(t)$, as $t \to \infty$, exists in R^n. The space $C_\ell(R_+, R^n)$ is a closed subspace of $BC(R_+, R^n)$, but $BC(R_+, R^n)$ is not closed in $C(R_+, R^n)$.

The main concern of this paper is finding adequate conditions on the data, more precisely on the operators L and V, such that the existence of solution to the problem (1), (2) is guaranteed (on the semi-axis R_+).

If we succeed to prove the existence of a solution to (1), (2), then we can draw some conclusions about its asymptotic behavior if we can show that it belongs to one of the function spaces we have mentioned above (like $BC(R_+, R^n)$, $C_\ell(R_+, R^n)$, or other function spaces).

2 An Auxiliary Result

We shall briefly discuss in this section a connection between causal operators on function spaces and classical/integral operators of Volterra type. For a more detailed discussion and further references, we send the reader to [1; Ch. 2 and Ch. 4], with more references.

The result we need is simply expressed by the formula

$$\int_0^t (Vx)(s)\, ds = \int_0^t K(t,s)x(s)\, ds, \quad t \in R_+, \tag{4}$$

where V stands for a linear causal operator on the space $C(R_+, R^n)$, and $K(t, s)$ denotes a matrix kernel of type $n \times n$.

More precisely, the result described by (4), states that for each V, there exists a measurable kernel $K(t, s)$, rendering the service described by (4). The formula (4) holds true for every $x \in C(R_+, R^n)$, and even in the most general case, $x \in L_{loc}(R_+, R^n)$. We notice that $K(t, s)$ must not be continuous. In order to assure the inclusion

$$\int_0^t K(t,s)x(s)\, ds \in C(R_+, R^n), \tag{5}$$

for each $x \in C(R_+, R^n)$, it suffices to deal with a locally integrable $K(t,s)$, $(t,s) \in \Delta = \{(t,s)\colon\ 0 \le s \le t\}$ satisfying also the condition

$$\lim_{h \to 0} \left(\int_0^t |K(t+h,s) - K(t,s)|\, ds + \int_t^{t+h} |K(t+h,s)|\, ds \right) = 0 \tag{6}$$

for each $t \in R_+$.

Let us notice that condition (6) on $K(t,s)$ is implied by (4), and the fact that V is a linear operator acting on $C(R_+, R^n)$. Hence, it takes continuous maps into continuous ones.

The kernel $K(t,s)$ from (4) automatically verifies other properties, if it does satisfy extra conditions. For instance, one frequently encountered property is

$$\sup \left\{ \int_0^t |K(t,s)|\, ds\colon \quad t \in R_+ \right\} < \infty, \tag{7}$$

is implied by the requirement that the subspace $BC(R_+, R^n)$ must be left invariant by the operator V.

In general, the connection between the operator V, and the properties of the kernel $K(t,s)$, is not always easy to be established. The problem of clarifying such connections is of great significance for applications.

For instance, an open problem is to establish the conditions on V, such that the associated kernel $K(t,s)$ admits a resolvent kernel $\widetilde{K}(t,s)$. See [3] for some discussion in this regard.

We shall formulate conditions for V, by means of the relationship (4). In other words, by imposing the adequate conditions on the associated kernel $K(t,s)$. In special situations, when V is chosen in a classical form, the connection may appear more transparent.

3 Equation (1) and Its Equivalent Forms

Let us return to equation (1) and make a few remarks that will help simplifying the coming considerations.

First, it is obvious that an additive constant to the operator L, i.e. a constant n-vector, does not change the equation (1). Hence, without loss of generality, we can assume

$$(Lx)(0) = \theta \in R^n, \tag{8}$$

for any x in the space $C(R_+, R^n)$. This property imposed on the causal operator L has been called by L. Neustadt [15], the "fixed initial value" property, and its significance in dealing with existence problems for (1) has been illustrated.

In case of classical Volterra operator

$$(Vx)(t) = f(t) + \int_0^t K(t,s,x(s))\, ds,$$

one obtains $(Vx)(0) = f(0) = \text{const}$, for any $x \in C(R_+, R^n)$, or in another underlying space.

As mentioned above we can substitute θ in (8) by any constant $c \in R^n$ without changing the equation.

By integrating both sides of (1) from 0 to $t > 0$ we obtain the functional differential equation

$$\dot{x}(t) - (Lx)(t) = v^0 + \int_0^t (Vx)(s)\, ds, \tag{9}$$

if we take into account (2) and (8).

The following equation is related to the equation (9),

$$\dot{x}(t) - (Lx)(t) = f(t), \quad t \in R_+. \tag{10}$$

As we know from [3, 7], the Cauchy problem for (10) with $x(0) = x^0 \in R^n$ can be represented in case of linear and continuous operator L on $C(R_+, R^n)$, by an integral formula, involving the Cauchy operator associated to L. It is sort of variation of parameters formula (Lagrange), and it looks

$$x(t) = X(t,0)\, x^0 + \int_0^t X(t,s)\, f(s)\, ds, \quad t \in R_+, \tag{11}$$

for any $f \in C(R_+, R^n)$. The Cauchy function (or kernel) $X(t,s)$ is defined by the formula

$$X(t,s) = I + \int_s^t \widetilde{K}_0(t,u)\, du, \tag{12}$$

for each $(t,s) \in \Delta$,

$$\Delta = \{(t,s) : \quad 0 \le s \le t\}, \tag{13}$$

where $\widetilde{K}_0(t,s)$ is the resolvent kernel corresponding to $K_0(t,s)$, $(t,s) \in \Delta$, from the representation

$$\int_0^t (Lx)(s)\, ds = \int_0^t K_0(t,s)\, x(s)\, ds, \quad t \in R_+. \tag{14}$$

In (14), $K_0(t,s)$ is measurable only, but the existence of $\widetilde{K}_0(t,s)$ is assured, for instance, if we accept the condition

$$K_0(t,s) \in L^\infty_{loc}(\Delta,\, \mathcal{L}(R^n, R^n)). \tag{15}$$

Let us apply formula (11) to the equation (9). We are led to the functional equation

$$x(t) = X(t,0)\, x^0 + \int_0^t X(t,s)\, v^0 ds + \int_0^t X(t,s) \int_0^s (Vx)(u)\, du\, ds. \tag{16}$$

Since we already assumed L to be linear, there results that (16), which is an equivalent equation to the problem (1), (2), is also linear when V is linear. Otherwise, it is a nonlinear functional equation for $x(t)$.

In case V is a linear operator acting on $C(R_+, R^n)$, we can use the representation (4), and (16) leads to another equivalent form of the problem (1), (2):

$$x(t) = X(t,0)x^0 + \int_0^t X(t,s)v^0\, ds + \int_0^t X(t,s) \int_0^s K(s,u)x(u)\, du\, ds.$$

Interchanging the order of integration in the double integral, we obtain the functional differential equation

$$x(t) = X(t,0)x^0 + \int_0^t X(t,s)v^0\, ds + \int_0^t \left(\int_u^t X(t,s)K(s,u)\, ds \right) x(u)\, du,$$

which can be rewritten as

$$x(t) = f(t) + \int_0^t K_1(t,u)x(u)\,du, \quad t \in R_+, \tag{17}$$

with

$$f(t) = X(t,0)\,x^0 + \int_0^t X(t,s)v^0\,ds, \quad t \in R_+, \tag{18}$$

and

$$K_1(t,u) = \int_u^t X(t,s)K(s,u)\,du, \quad 0 \le u \le t. \tag{19}$$

The equation (17) will be discussed in detail in the next section, and the existence result will be applied to the problem (1), (2).

Another case will be considered, when the operator V is not necessarily linear, but it is Lipschitz continuous.

4 The Linear Case: Equation (17)

We have to examine the function (18) and the kernel $K_1(t,u)$ to see if we can construct a solution in $C(R_+, R^n)$.

First, the function $f(t)$ from (18) is the solution of the functional differential equation $\dot{x}(t) - (Lx)(t) = v^0$, with the initial condition $x(0) = x^0$. Hence, $f(t)$ is a continuously differentiable function on R_+, with values in R^n.

Second, the kernel $K_1(t,u)$ given by (19) is a locally bounded function on Δ. More precisely, we can infer

$$K_1(t,s) \in L^\infty_{loc}(\Delta,\, \mathcal{L}(R^n, R^n)). \tag{20}$$

Indeed, we shall admit that $K(t,s)$ belongs to L^∞_{loc}, as mentioned above. Furthermore, formula (12) shows us that $X(t,s)$ is also locally bounded on Δ. This fact is a consequence of property that states: any kernel $K(t,s)$, which is locally bounded on Δ, admits a resolvent $\widetilde{K}(t,s)$, which is also locally bounded on Δ.

Therefore, the integral equation (17), whose kernel $K_1(t,s)$ satisfies (20), admits a resolvent kernel $\widetilde{K}_1(t,s)$, locally bounded on Δ, while its unique solution is represented by the resolvent formula

$$x(t) = f(t) + \int_0^t \widetilde{K}_1(t,s)\,f(s)\,ds, \quad t \in R_+, \tag{21}$$

for any function $f \in L^\infty_{loc}(R_+, R^n)$. In particular, (21) holds true when $f(t)$ is given by the formula (18).

In summarizing the discussion carried out above, we can state the following existence result for the problem (1), (2).

Theorem 4.1 *Consider the neutral functional differential equation (1), with the initial conditions (2). Assume the following conditions are satisfied:*

(i) *the operators L and V are linear, continuous and causal, acting on the space $C(R_+, R^n)$;*

(ii) *the kernels* $K(t,s)$ *and* $K_0(t,s)$*, occuring in the representations (4) and (14) are locally bounded on* Δ*, defined by (13).*

Then, there exists a unique solution $x(t)$*,* $t \in R_+$*, of the problem (1), (2), for arbitrary initial data* $x^0, v^0 \in R^n$*. This solution is continuously differentiable on* R_+*.*

The proof is immediate if we rely on the discussion preceding the statement of Theorem 1, the equivalence of (1), (2) with the equation (17) being the key ingredient.

Remark 4.1 The condition (6) is not the only condition that can be derived from the fact that V is acting on $C(R_+, R^n)$.

Indeed, from (6) we read

$$\int_0^t K(t,s)x(s)\,ds \in AC_{loc}(R_+, R^n), \tag{22}$$

for every $x \in C(R_+, R^n)$. The space AC_{loc} in (22) is a subspace of $C(R_+, R^n)$, and the inclusion

$$\int_0^t K(t,s)x(s)\,ds \in C(R_+, R^n), \quad t \in R_+, \tag{23}$$

tells us less than (22). Nevertheless, we prefer to use (23) instead of (22) for simplicity. For example, (23) implies

$$\lim_{h\to 0} \int_0^t |K(t+h,s) - K(t,s)|ds = 0 \quad t \in R_+. \tag{24}$$

Remark 4.2 Considering the resolvent formula (21) for the solution of equation (17), we can obtain more information about the solution of (1), (2), making extra assumptions on the kernel $\widetilde{K}_1(t,s)$. This kernel is determined, as shown by (19), by the properties of the operators L and V.

For instance, if we assume that $\widetilde{K}_1(t,s)$ satisfies the condition

$$\int_0^t |\widetilde{K}_1(t,s)|\,ds \le M < \infty, \quad t \in R_+, \tag{25}$$

and also

$$|X(t,0)| + \int_0^t |X(t,s)|\,ds \le N < \infty, \quad t \in R_+, \tag{26}$$

then the solution of the problem (1), (2) will verify the inclusion

$$x(t) \in L^\infty(R_+, R^n), \quad x^0, v^0 \in R^n. \tag{27}$$

The Proof follows immediately from the formulas (18) and (21).

Further properties of the solution can be obtained by imposing various types of estimates on the kernels $K_1(t,s)$ or $\widetilde{K}_1(t,s)$, as well as on $f(t)$.

The main problem is to establish the connection between the properties of the operators L and V, and the kernels occurring in the representations (4) and (14). This will be discussed in forthcoming papers.

5 A Nonlinear Case: Equation (16)

We shall rewrite equation (16) in the form

$$x(t) = f(t) + \int_0^t X(t,s) \int_0^s (Vx)(u)\, du\, ds, \quad t \in R_+, \tag{28}$$

where $f(t)$ is given by (18). Equation (28), with $f(t)$ defined by (18), is equivalent to our problem (1), (2). This is a functional integral equation and we shall treat it by the classical method of iteration/successive approximations. This approach will lead to an existence and uniqueness result in the space $C(R_+, R^n)$. Of course, the operator V is assumed to be acting on this space.

In order to simplify somewhat the procedure, we shall adopt a hypothesis which is part of the assumption (25) above. This hypothesis concerns only the linear operator L, which fully determines the Cauchy kernel $X(t,s)$, $0 \le s \le t$.

Namely, we assume in this section that

$$\int_0^t |X(t,s)|\, ds \le M < \infty, \quad (t,s) \in \Delta, \tag{29}$$

and we shall limit our consideration in regard to equation (28), only to those operators L for which (29) is satisfied.

Concerning the operator V, acting on the same space $C(R_+, R^n)$, we shall assume it verifies the Lipschitz type condition

$$|(Vx)(t) - (Vy)(t)| \le \lambda(t)\, |x(t) - y(t)|, \quad t \in R_+, \tag{30}$$

for any $x, y \in C(R_+, R^n)$. We also assume that $\lambda(t)$ is a nonnegative nondecreasing map from R_+ into itself.

In order to prove the existence and uniqueness of a solution to (28), we construct the sequence of successive approximations $\{x_k(t)\colon\ k \ge 0\}$, by letting $x_0(t) = f(t)$, and

$$x_{k+1}(t) = f(t) + \int_0^t X(t,s) \int_0^s (Vx_k(u))\, du\, ds, \tag{31}$$

for $k \ge 1,\ t \in R_+$.

We shall prove now that the sequence of successive approximations converges in $C(R_+, R^n)$. This means that the sequence converges uniformly on each bounded interval $[0,T] \subseteq R_+$. The limit of this sequence

$$x(t) = \lim_{k\to\infty} x_k(t), \quad t \in R_+, \tag{32}$$

will constitute the solution of our problem. As usual, if we subtract side by side the relationship (31) and the one corresponding to k instead of $k+1$, we find the following recurrent relation, valid for $k \ge 1$ and $t \in R_+$:

$$x_{k+1} - x_k(t) = \int_0^t X(t,s) \int_0^s [(Vx_k)(u) - (Vx_{k-1})(u)]\, du\, ds. \tag{33}$$

Taking into account (29) and (30), we obtain from (33) the following recurrent inequality:

$$|x_{k+1}(t) - x_k(t)| \le M \sup_{0\le s\le t} \int_0^s \lambda(u)|x_k(u) - x_{k-1}(u)|\, du.$$

The above inequality leads immediately to

$$|x_{k+1}(t) - x_k(t)| \leq M \int_0^t \lambda(s) \sup_{0\leq u\leq s} |x_k(u) - x_{k-1}(u)|\, du. \tag{34}$$

Let us denote

$$y_k(t) = \sup_{0\leq s\leq t} |x_k(s) - x_{k-1}(s)|, \tag{35}$$

and rewrite (34) in the form

$$y_k(t) \leq M \int_0^t \lambda(s) y_k(s)\, ds, \quad k \geq 1. \tag{36}$$

We had to keep in mind the fact that $\sup\{|x_k(t) - x_{k-1}(t)|\colon\ t \in [0,T]\}$ is nondecreasing in T.

Now by induction, the recurrent inequality (36) leads to, if one assumes $y_1(t) \leq A$ on the interval $[0,T]$, $T > 0$ arbitrary,

$$y_{k+1}(t) \leq A\, \frac{M^k}{k!} \left(\int_0^t \lambda(u)\, du \right)^k, \quad t \in [0,T]. \tag{37}$$

The inequality (37) obviously implies the uniform convergence of the sequence of successive approximations, on any finite interval of R_+. Therefore, we have

$$\lim_{k\to\infty} x_k(t) = x(t) \in C(R_+, R^n). \tag{38}$$

The function $x(t)$ defined by (38) is a solution of equation (17), and this equation is equivalent to the problem (1), (2). While (38) shows that $x(t)$ is a continuous solution, it has actually better regularity properties, as stipulated in section 3.

Let us now formulate the main result of this section, related to our basic problem (1), (2).

Theorem 5.1 *Consider the initial value problem (1), (2), or equivalently the functional integral equation (17), under the following assumptions:*

(i) *The operator L is a linear continuous operator on the space $C(R_+, R^n)$.*

(ii) *The operator V is also acting on the space $C(R_+, R^n)$, and verifies the Lipschitz condition (30), with $\lambda(t)$ nondecreasing on R_+.*

Then, there exists a unique solution $x(t) \in C(R_+, R^n)$, of (1), (2), or (17), and it is continuously differentiable on R_+, as well as $\dot{x}(t) - (Lx)(t)$.

Remark 5.1 The uniqueness is proven in the same way we have shown the convergence of the successive approximations, and using estimates like (37).

Remark 5.2 The function $f(t)$ given by (18) is continuously differentiable. It would suffice to be just continuous.

Once we know the solution of our problem does exist, we can think of obtaining further properties, of asymptotic nature.

Namely, we shall drop the assumption (25) on the resolvent kernel $\widetilde{K}_1(t,s)$, and impose other conditions that can be verified more directly, on the operator V. These conditions are

$$(V\theta)(t) \equiv 0, \quad \lambda(t) \in L^1(R_+, R), \tag{39}$$

and they will help us to recognize the property of boundedness for the solution $x(t) \in BC(R_+, R^n)$.

Indeed, we see that the conditions of Theorem 5.1 are verified if we accept (39) and the Lipschitz condition with $\lambda(t)$ instead of the Lipschitz constant. But if we define $\tilde{\lambda}(t) = \sup\{\lambda(s) : \ 0 \le s \le t\}$, we find a function providing as $\lambda(t)$ does. Obviously, (39) implies $|(Vx)(t)| \le \lambda(t)|x(t)| \le \tilde{\lambda}(t)|x(t)|$. Hence, on behalf of Theorem 5.1 we have assured the existence and uniqueness of the solution.

We shall prove that the solution is actually in $BC(R_+, R^n)$, if we make the extra assumption (26) on $X(t,s)$.

From (26), (28) and (29), we obtain the integral inequality

$$|x(t)| \le N + M \int_0^t |(Vx)(s)|\, ds, \quad t \in R_+, \tag{40}$$

$x(t)$ being the solution in $C(R_+, R^n)$ for our problem. The inequality (40) and Lipschitz condition imply

$$|x(t)| \le N + M \int_0^t \lambda(s)\, |x(s)|\, ds, \quad t \in R_+. \tag{41}$$

The inequality (41) is of Gronwall type, and yields

$$|x(t)| \le M \exp\left(M \int_0^\infty \lambda(s)\, ds \right), \quad t \in R_+,$$

which shows that $x(t) \in BC(R_+, R^n)$, as it follows from second condition (39).

6 Some Final Remarks

As seen above, the existence of the resolvent kernel is very helpful. Besides the case $K_0(t,s) \in L^\infty_{loc}(\Delta, \mathcal{L}(R^n, R^n))$, the resolvent kernel does exist in other cases. For instance, when $K_0(t,s) \in L^2_{loc}(\Delta, \mathcal{L}(R^n, R^n))$. A parallel investigation of problem (1), (2) could be conducted in this case. For the general framework see [16].

Several problems of asymptotic behavior of solutions can be treated, with extra assumptions, in the framework used in this paper. For instance, looking for solutions in the space $C_\ell(R_+, R^n)$.

We shall close these remarks with a dynamical interpretation of equation (28), which is equivalent to our problem (1), (2), when the function $f(t)$ is chosen in the form (18).

Equation (17) describes the working of a feedback dynamical system, in which the linear plant is described by the equation

$$x(t) = f(t) + \int_0^t X(t,u)c(u)\, du, \tag{42}$$

where $c(u)$ represents the input, $c(u) \in E(R_+, R^n)$, while the feedback action is given by

$$c(t) = \int_0^t (Vx)(s)\,ds, \quad t \in R_+. \tag{43}$$

The information about the asymptotic behavior of the solution of (28) can be directly related to the motion of the dynamical system. The mathematical and engineering literatures are providing numerous applications of such systems. Some examples and further discussion, with ample references to literature, can be found in [1, 2, 10, 11, 14].

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A Robust Detector for a Class of Uncertain Systems

Maher Hammami [1*] and Ines Ellouze [2]

[1] *Département de Physique, Faculté des Sciences de Sfax*
Rte de Soukra Km 3.5, B.1171 - 3000 Sfax-Tunisie
Fax: (216) 74 274 437
[2] *Département de Mathématiques, Faculté des Sciences de Sfax*

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Abstract: This paper studies the problem of output feedback stabilization of a class of uncertain systems. We construct a robust detector which provides an approximation of the state of the system. The state trajectory control by state observation for a class of uncertain systems based on output feedback is treated, where the nominal system is linear and the uncertainties are bounded. This work is based on Lyapunov techniques. Furthermore, a numerical example is given to illustrate the applicability of our main result.

Keywords: *Uncertain systems; state observation; output-controller; detector.*

Mathematics Subject Classification (2000): 93D09, 34D20.

1 Introduction

The problem of state trajectory control for nonlinear systems by output feedback is addressed by several authors ([1]–[10]) using several basic methods of studying the stability and constructing stabilizing output controllers.

In this paper, we treat this problem for a class of uncertain systems. The perturbation term could result from errors in modeling the nonlinear system, aging of parameters or uncertainties. In a typical situation, we do not know the uncertainties but we know some information about it. We can no longer study stability of the origin as an equilibrium point, nor should we expect the solution of the uncertain systems to approach the origin as t tends to infinity. The best we can hope that, if the uncertainties are bounded by a small term in some sense, then the solution will be ultimately bounded by a small bound

* Corresponding author: hammami_maher@yahoo.fr

for sufficiently large t. Under some conditions we construct a robust detector (dynamical system which is expected to produce an estimation of the state on the hole space except on a small neighborhood of the origin) as the one introduced by Vidyasager [11]. We study the state trajectory control for non-linear system by output feedback. We obtain Global Uniform Ultimate Boundedness (GUUB) trajectory (see [12]) for the state of the error equation.

Consider the state space model

$$\begin{cases} \dot{x} &= A(\cdot)x + B(\cdot)u, \\ y &= C(\cdot)x, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, $y \in \mathbb{R}^p$, $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ depend on some parameter and $(n \times n)$, $(n \times q)$ and $(p \times n)$ matrices respectively. We shall assume that the dimension of the state model is finite. We consider throughout this paper specifically perturbations of the state space from of the plant dynamics (i.e., perturbations of the $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ matrices). Let A_0, B_0 and C_0 be the linearized nominal model of the plant. The matrices $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ can be factored as follows

$$A(\cdot) = A_0 + \Delta A,$$

$$B(\cdot) = B_0 + \Delta B,$$

$$C(\cdot) = C_0 + \Delta C.$$

We suppose here the exact knowledge of the state space matrices $(A_0,\ B_0,\ C_0)$. The elements of the matrix A_0 are $\{a_{ij}\}$ while the elements of the matrix $A(\cdot)$ are $\{a_{ij} + \delta_{ij}\}$. In the absence of nonlinearities, the problem is reduced to the linear one.

$$\begin{cases} \dot{x} &= A_0 x + B_0 u, \\ y &= C_0 x. \end{cases} \tag{2}$$

Uncertain systems are an important class of nonlinear systems, several authors are interested to study this kind of systems. In [13] and [14], the authors studied this class of system when the nonlinear part is of the form $Ew + \sigma$, the noise w is described in a general state space form and it also includes the case of state dependent noise. The Ew factor may represent a stochastic parameter variation of the system matrix A while σ represents an external additive perturbation.

2 System and Definitions

Let consider the system (1) which can be described by the following state equations

$$\begin{cases} \dot{x} &= A_0 x + B_0 u + \Delta A x + \Delta B u, \\ y &= C_0 x + \Delta C x, \end{cases} \tag{3}$$

and the following detector

$$\dot{\hat{x}} = A_0 \hat{x} + B_0 u - L(C_0 \hat{x} - y) + \Delta A \hat{x} + \Delta B u.$$

Let $e = \hat{x} - x$. The error equation is given locally by

$$\dot{e} = (A_0 - LC_0)e + o(e).$$

Since the terms $\|\Delta Ax\|$, $\|\Delta Bu\|$ and $\|\Delta Cx\|$ are locally bounded, then these dynamics are locally exponentially stable provided that the pair (A_0, C_0) is detectable. In [15], [16] the authors studied this class of systems and constructed a global detector in the presence of nonlinear perturbation.

In the general case, we consider the perturbed system (3) which can be described by the following state equations

$$\begin{cases} \dot{x} = & F(t,x,u) & = A_0 x + B_0 u + \Delta f(t,x,u), \\ y = & h(t,x) & = C_0 x + \Delta h(t,x), \end{cases} \tag{4}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^q,\ y \in \mathbb{R}^p,\ A_0,\ B_0$ and C_0 are known $(n \times n),\ (n \times q),\ (p \times n)$ constant matrices respectively and $\Delta f(t,x,u)$, $\Delta h(t,x)$ are locally Lipschitz continuous represents of the uncertainties in the plant. For our case, in the presence of uncertainties, we give a definition of detectability, where we introduce the notion of a global detector and we will study the state observation law for a class of uncertain systems in the GUUB trajectory sense.

Consider the system

$$\begin{cases} \dot{x} & = & f(t,x), \\ y & = & h(t,x), \end{cases} \tag{5}$$

where $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^q$ is the control and $y \in \mathbb{R}^p$ is the output of the system. The functions $f : [0,+\infty[\times\mathbb{R}^n \to \mathbb{R}^n$ respectively $h : [0,+\infty[\times\mathbb{R}^n \to \mathbb{R}^p$ are pieceswise continuous in t and globally Lipschitz in x on $[0,+\infty[\times\mathbb{R}^n$.

We now introduce the notions of uniform boundedness and uniform ultimate boundedness of a trajectory of (5) (see [12]).

Definition 2.1 The system (5) is uniformly bounded when

- for all $R_1 > 0$, there exists a $R_2 = R_2(R_1) > 0$ such that for all $x_0 \in \mathbb{R}^n$, for all t_0 and for all $t \geq t_0$

$$\|x_0\| \leq R_1 \Longrightarrow \|x(t)\| \leq R_2.$$

Definition 2.2 The system (5) is uniformly ultimately bounded when

- there exists a $R > 0$ such that for all $R_1 > 0$, there exists a $T = T(R_1) > 0$ such that for all $x_0 \in \Re^n$, for all t_0 and for all $t \geq t_0 + T$

$$\|x_0\| \leq R_1 \Longrightarrow \|x(t)\| \leq R.$$

The above definition means that we have the ultimately bound of the trajectory uniformly on t_0. The classical theorem of Lyapunov proves uniform asymptotic stability of the equilibrium point $x = 0$ of a dynamical system $\dot{x}(t) = f(x(t),t)$ when there exists a positive definite and decrescent Lyapunov function $V(x,t)$ whose derivative $\dot{V}(x,t)$ along the solutions of the system is negative definite. When there exists a $R_V > 0$ such that the derivative $\dot{V}(x,t)$ along the solutions of the system is negative for x with $\|x\| > R_V > 0$.

Definition 2.3 The system (5) is GUUB solution if $\dot{V}$ satisfies the following estimation:

$$\dot{V}(x(t)) \leq -\eta V(x(t)) + r \tag{6}$$

with $\eta > 0$ and $r > 0$.

Remark 2.1 If equation (6) holds then the state of (3) satisfies:

$$\| x(t) \| \quad \leq \quad \| x(t_0) \| \, e^{-\eta(t-t_0)} + \frac{r}{\eta}, \ \forall \ t \geq t_0.$$

The problem is to design a continuous detector with input $y(t)$ such that the estimates denoted by $\hat{x}(t)$ converge to $x(t)$ in the ultimate bounded sense (as in the Definition 2.3).

Definition 2.4 (Robust detector). A system

$$\dot{\hat{x}} = G\left(t, \hat{x}, y, u\right)$$

is called a robust detector for (3) if for all input signals u,

$$\forall \ \|\hat{x}(t_0) - x(t_0)\| \ \in \ \Re^n \backslash B(0, \frac{r}{\eta})$$

one has

$$\|\hat{x}(t) - x(t)\| \quad \leq \quad \lambda_1 \|\hat{x}(t_0) - x(t_0)\| e^{-\eta(t-t_0)} + \frac{r}{\eta}, \ \ \forall \ t \geq t_0.$$

$B(0, \frac{r}{\eta})$ denotes the ball of radius $\frac{r}{\eta} > 0$ with $\lambda_1 > 0, r > 0$ and $\eta > 0$. Note that the state of the error equation converges to the ball $B(0, \frac{r}{\eta})$ when t goes to infinity.

3 Robust Detector

We now highlight the major assumptions, with regard to the system given by (3) that are used in the observer stability proof.

$(\mathcal{A}_1)$ The pair (A_0, C_0) is observable, then there exists a matrix L such that the eigenvalues of $(A_0 - LC_0)$ are in the open left-half plane [17]. For all definite positive symmetric matrix Q there exists a definite positive symmetric matrix P such that:

$$(A_0 - LC_0)^T P + P(A_0 - LC_0) = -Q.$$

$(\mathcal{A}_2)$ There exists a function ϕ where $\phi(.,.,.) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^p$, such that

$$P\Delta f(t, x, u) = C_0^T \phi(t, x, u),$$

where P is the unique positive definite solution to the Lyapunov equation which is given in $(\mathcal{A}_1)$.

$(\mathcal{A}_3)$ There exits a positive scalar function $\delta_1(t)$ such that

$$\|\phi(t, x, u)\| \leq \delta_1(t),$$

where $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

$(\mathcal{A}_4)$ There exists a function γ where $\gamma(.,.) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^p$, such that

$$PL\Delta h(t, x) = C_0^T \gamma(t, x),$$

where P is the unique positive definite solution to the Lyapunov equation which is given in $(\mathcal{A}_1)$.

$(\mathcal{A}_5)$ There exists a positive scalar function $\delta_2(t)$ such that

$$\|\gamma(t, x)\| \leq \delta_2(t),$$

where $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

Theorem 3.1 *If the assumption* $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$, $(\mathcal{A}_4)$ *and* $(\mathcal{A}_5)$ *hold, then the system*

$$\dot{\hat{x}} = A_0\hat{x} + B_0 u + \varphi(t,\hat{x},y,u) - L(C_0\hat{x} - y),$$

where

$$\varphi(t,\hat{x},y,u) = -\frac{P^{-1}C_0^T(C_0\hat{x}-y)\delta(t)^2}{\|C_0 e\|\delta(t) + r_0} \tag{7}$$

with $r_0 > 0$ *and* $\delta(t) = \delta_1(t) + \delta_2(t)$, *is a robust detector for the system (4).*

Proof Let consider the following Lyapunov function $V(e) = e^T P e$ as in the $(\mathcal{A}_1)$. The derivative of this function along the trajectory of the closed-loop system by the output feedback y, or just along the error equation and using equation (4) and (7)

$$\dot{e} = (A_0 - LC_0)e - \Delta f(t,x,u) + \varphi(t,\hat{x},y,u) + L\Delta h(t,x)$$

is given by

$$\dot{V}(e) = -e^T Q e - 2e^T P\Delta f(t,x,u) + 2e^T P\varphi(t,\hat{x},y,u) + 2e^T PL\Delta h(t,x)$$

and using equation (7), $(\mathcal{A}_2)$ and $(\mathcal{A}_4)$

$$\begin{aligned}
\dot{V}(e) &= -e^T Q e - 2e^T C_0^T\phi(t,x,u) - 2e^T P\frac{P^{-1}C_0^T(C_0\hat{x}-y)\delta(t)^2}{\|C_0 e\|\delta(t) + r_0} \\
&\quad +2e^T C_0^T\gamma(t,x) \\
&= -e^T Q e - 2e^T C_0^T\phi(t,x,u) - 2\frac{e^T C_0^T C_0 e\delta(t)^2}{\| C_0 e \| \delta(t) + r_0} + 2\frac{e^T C_0^T\Delta h(t,x)\delta(t)^2}{\| C_0 e \| \delta(t) + r_0} \\
&\quad +2e^T C_0^T\gamma(t,x).
\end{aligned}$$

Since

$$\lambda_{min}(Q) \parallel e \parallel^2 \le\ e^T Q e\ \le\ \lambda_{max}(Q) \parallel e \parallel^2, \tag{8}$$

where $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the minimum and maximum eigenvalues of the matrix A and using $(\mathcal{A}_3)$ and $(\mathcal{A}_5)$, one gets

$$\begin{aligned}
\dot{V}(e) &\le -\lambda_{min}(Q)\|e\|^2 + 2\|C_0 e\|\delta_1(t) - 2\frac{\|C_0 e\|^2\delta(t)^2}{\|C_0 e\|\delta(t) + r_0} + 2\frac{\|C_0 e\|\|\Delta h(t,x)\|\delta(t)^2}{\|C_0 e\|\delta(t) + r_0} \\
&\quad +2\|C_0 e\|\delta_2(t)
\end{aligned}$$

with $\delta(t) = \delta_1(t) + \delta_2(t)$,

$$\begin{aligned}
\dot{V}(e) &\le -\lambda_{min}(Q)\|e\|^2 + 2\|C_0 e\|\delta(t) - 2\frac{\|C_0 e\|^2\delta(t)^2}{\|C_0 e\|\delta(t) + r_0} + 2\frac{\|C_0 e\|\|\Delta h(t,x)\|\delta(t)^2}{\|C_0 e\|\delta(t) + r_0} \\
&\le -\lambda_{min}(Q)\|e\|^2 + 2r_0\frac{\|C_0 e\|\delta(t)}{\|C_0 e\|\delta(t) + r_0} + 2\delta(t)\|\Delta h(t,x)\|\frac{\|C_0 e\|\delta(t)}{\|C_0 e\|\delta(t) + r_0} \\
&\le -\lambda_{min}(Q)\|e\|^2 + (2r_0 + 2\delta(t)\|\Delta h(t,x)\|)\ \frac{\|C_0 e\|\delta(t)}{\|C_0 e\|\delta(t) + r_0}.
\end{aligned}$$

Since,

$$\frac{\|C_0 e\|\delta(t)}{\|C_0 e\|\delta(t) + r_0} \ < \ 1, \quad \|\Delta h(t,x)\| < \frac{\|C_0\|\delta_2(t)}{\|PL\|},$$

we obtain

$$\begin{aligned} \dot{V}(e) &\leq -\lambda_{min}(Q)\|e\|^2 + 2r_0 + 2\frac{\|C_0\|\delta_2(t)}{\|PL\|}\delta(t) \\ \dot{V}(e) &\leq -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}V(e) + r \end{aligned}$$

with

$$r = 2r_0 + 2\frac{\|C_0\|\delta_2(t)}{\|PL\|}\delta(t).$$

So

$$\dot{V}(e) \leq -\eta V(e) + r$$

with

$$\eta = \frac{\lambda_{min}(Q)}{\lambda_{max}(P)}. \tag{9}$$

From the above Remark 2.1 one obtains the following estimation

$$\| \, V(e(t)) \, \| \quad \leq \quad \| \, V(e(t_0)) \, \| \, e^{-\eta(t-t_0)} + \frac{r}{\eta}$$

so,

$$\lambda_{min}(P)\|e(t)\|^2 \quad \leq \quad \lambda_{max}(P)\|e(t_0)\|^2 e^{-\eta(t-t_0)} + \frac{r}{\eta}.$$

Hence,

$$\|e(t)\| \quad \leq \quad \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}\|e(t_0)\|e^{-\frac{\eta}{2}(t-t_0)} + \sqrt{\frac{r}{\lambda_{min}(P)\eta}}.$$

Therefore, $e(t)$ converges to the ball $B(0, \sqrt{\frac{r}{\lambda_{min}(P)\eta}})$ in the ultimate bounded sense.□

Remark 3.1 Note that, if we take $r_0 = r_0(t)$ with $r_0(t)$ going to zero when t tends to infinity and $\delta \longrightarrow 0$ when $t \rightarrow +\infty$, the trajectory tends to the origin exponentially when $t \rightarrow +\infty$.

Next, we consider the system (4) under the condition that the uncertainties are bounded. When the states are not available the usual technique is to build an observer which gives an approximation of the state. Many authors studied the problem of the conception of the observer. For the concept of observer, we aim at simplifying the design of this system by exploiting the linear form of the nominal system. We first introduce the following definition as in [18] and [19].

Definition 3.1 A practical exponential observer for (4) is a dynamical system which has the following form:

$$\dot{\hat{x}} = F(t, \hat{x}, u) - L(C\hat{x} - y), \tag{10}$$

where L is the gain matrix and the origin of the error equation with $e = \hat{x} - x$, which is given by

$$\dot{e} = F(t, \hat{x}, u) - F(t, x, u) - LCe + L\Delta h(t, x) \tag{11}$$

is globally practically exponentially stable, it means that it is globally uniformly practically asymptotically stable and the following estimation holds:

$$\|e(t)\| \leq \lambda_1(\|e(t_0)\|)\, e^{-\lambda_2(t-t_0)} + r, \quad \forall\, t \geq t_0$$

with $\lambda_1,\ \lambda_2,\ r > 0$.

Note that, the origin $x = 0$ may not be an equilibrium point of the system (4). We can no longer study stability of the origin as an equilibrium point nor should we expect the solution of the system to approach the origin as $t \longrightarrow \infty$. The inequality given in Remark 2.1 implies that $x(t)$ will be ultimately bounded by a small bound $r > 0$, that is, $\|x(t)\|$ will be small for sufficiently large t. If r can be replaced by a smooth map $r(t)$ as a function of t which tends to zero as t tends to $+\infty$, the ultimate bound approaches zero. This can be viewed as a robustness property of convergence to the origin provided that F satisfies $F(t, 0, 0) = 0,\ \forall\ \ t \geq 0$, which is supposed in such a way the origin becomes an equilibrium point.

We consider the system (4) satisfying the assumption $(\mathcal{A}_1)$ and the following one

$(\mathcal{A}_6)$ There exist positive constants M_1 and M_2, such that for all $t \geq 0$

$$\begin{aligned} \|\Delta f(t, x)\| &\leq M_1, \\ \text{and} \qquad & \\ \|\Delta h(t, x)\| &\leq M_2. \end{aligned} \tag{12}$$

To design an observer, we shall consider the dynamical system

$$\dot{\hat{x}} = A\hat{x} + Bu + \Delta f(t, \hat{x}) - L(C\hat{x} - y), \tag{13}$$

where L is the gain matrix, $\hat{x} \in \mathbb{R}^n$ is the state estimate of $x(t)$ in the sense that $e(t) = \hat{x}(t) - x(t)$ satisfies the following estimation,

$$\|e(t)\| \leq \|e(t_0)\| e^{-\lambda(t-t_0)} + r,\ \forall\ \ t \geq t_0.$$

Proposition. *Under assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_6)$ the system* (13) *is a practical exponential observer for the system* (4).

Indeed, we consider the error equation with $e = \hat{x} - x$

$$\dot{e} = \dot{\hat{x}} - \dot{x} = (A_0 - LC_0)e + \Delta f(t, \hat{x}) - \Delta f(t, x) + L\Delta h(t, x) \tag{14}$$

and the quadratic Lyapunov function candidate $V(e) = e^T Pe$ as in the proof of the theorem. Taking into account $(\mathcal{A}_6)$, the derivative of W along the trajectories of system

(2) is given by

$$\begin{aligned}
\dot{W}(t,e) &= \dot{e}^T Pe + e^T P\dot{e} \\
&= e^T \left[(A_0 - LC_0)^T P + P(A_0 - LC_0)\right] e + 2e^T P(\Delta f(t,\hat{x}) - \Delta f(t,x)) \\
&\quad + 2e^T PL\Delta h(t,x) \\
&= -e^T Qe + 2e^T P(\Delta f(t,\hat{x}) - \Delta f(t,x)) + 2e^T PL\Delta h(t,x) \\
&\leq -e^T Qe + 2\|e^T P\| \cdot \|\Delta f(t,\hat{x}) - \Delta f(t,x)\| + 2\|e^T PL\| \cdot \|\Delta h(t,x)\| \\
&\leq -e^T Qe + 2\|P\| \left(\|\Delta f(t,\hat{x})\| + \|\Delta f(t,x)\|\right) \|e\| + 2\|P\| \cdot \|L\| \cdot \|\Delta h(t,x)\| \cdot \|e\| \\
&\leq -e^T Qe + 4\|P\| \cdot M_1 \cdot \|e\| + 2\|P\| \cdot \|L\| \cdot M_2 \cdot \|e\| \\
&\leq -e^T Qe + \left(4\|P\| M_1 + 2\|P\| \cdot \|L\| \cdot M_2\right) \|e\| \\
&\leq -\lambda_{min}(Q)\|e\|^2 + M\|e\|
\end{aligned}$$

with

$$M = \left(4\|P\|M_1 + 2\|P\| \cdot \|L\| \cdot M_2\right).$$

Using (8), we get

$$\dot{V}(e) \quad \leq \quad -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} V(e) + M\|e\|$$

and using (9)

$$\dot{V}(e) \quad \leq \quad -\eta V(e) + M\|e\|.$$

Therefore,

$$\dot{V}(e) \ \leq \ -\eta V(e) + \frac{M}{\sqrt{\lambda_{min}(P)}} \sqrt{V(e)}. \tag{15}$$

Let $W(t) = \sqrt{V(t)}$. The derivative with respect time yields

$$\dot{W}(t) = \frac{\dot{V}(t)}{2W(t)}.$$

So,

$$\dot{W}(t) \ \leq \ -\frac{1}{2}\eta W(t) + \frac{1}{2}\frac{M}{\sqrt{\lambda_{min}(P)}}.$$

Using remark 2.1, one gets

$$\|W(t)\| \ \leq \ \|W(t_0)\| e^{-\frac{1}{2}\eta(t-t_0)} + \frac{M}{\eta\sqrt{\lambda_{min}(P)}}.$$

Since $W(t) = \sqrt{e^T(t)Pe(t)}$, it follows that

$$\|e(t)\| \ \leq \ \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \cdot \|e(t_0)\| e^{-\frac{\eta}{2}(t-t_0)} + \frac{M}{\eta\lambda_{min}(P)}.$$

We get an estimation as in (7). The origin of (14) satisfies an estimation as in Definition 3.1. Hence, we conclude that, the origin of system (10) is a practical exponential observer for the system (4). The solution converges to the ball $B(0, \frac{M}{\eta\lambda_{min}(P)})$. $\square$

Note that $\|e(t)\|$ can be small for sufficiently large t, if we take $M = M(t)$ such that

$$\lim_{t\to\infty} M(t) = 0.$$

4 Numerical Example

Consider the system

$$\begin{cases} \dot{x}_1 &= x_1 + x_2 + e^{-t}\sin x_1, \quad t \geq 0, \\ \dot{x}_2 &= -3x_2 + 2u, \\ y &= x_1 + x_2, \end{cases} \tag{16}$$

with $x = (x_1, x_2)^T \in \mathbb{R}^2$,

$$\begin{array}{lll} A_0 &=& \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0, \\ 2 \end{pmatrix}, \\ C_0 &=& \begin{pmatrix} 1 & 1 \end{pmatrix}, \qquad \Delta f(t,x) = e^{-t}\sin x_1. \end{array} \tag{17}$$

This system is observable with

$$L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} 2401 \\ -2283 \end{pmatrix}.$$

We get the following system

$$\dot{\hat{x}} = \begin{pmatrix} -2400 & -2400 \\ 2283 & 2280 \end{pmatrix}\hat{x} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}u + \begin{pmatrix} 2401 \\ -2283 \end{pmatrix}y + e^{-t}\sin\hat{x}_1 \tag{18}$$

with

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lambda_{min}(Q) = \lambda_{max}(Q) = 1,$$

and P is given in $(\mathcal{A}_1)$:

$$P = \begin{bmatrix} 6,3458 & -6,3456 \\ -6,3456 & 6,3538 \end{bmatrix}, \quad \lambda_{min}(P) = 0,0042, \quad \lambda_{max}(P) = 12,6954.$$

Let $\|P\| = \lambda_{max}(P) = 12,6954$. Hence $\eta = 0,0788$ and $\Delta f(t,x) \leq 1, \ \forall\ t,x$. Here $M_1 = 1$, $M_2 = 0$ and $M = 4 \cdot \|P\| \cdot M_1 = 50,7816$.

Therefore, system (18) is an observer which can be considered as a robust detector as the definition 2.4 and the trajectory of the error equation converges to the ball $B(0,r)$ with $r \simeq 153498$.

Conclusion

This paper deals with the problem of the output stabilisation for a class of uncertains systems. It is shown that an output controller can be constructed under some sufficient conditions and a robust detector can be designed which provides an estimation of the state. A numerical example in the plane is given to illustrate our main result.

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Aquifer Parameter Identification with Hybrid Ant Colony System

Shouju Li *, He Yu and Yingxi Liu

State Key Laboratory of Structural Analysis of Industrial Equipment,
Dalian University of Technology, Dalian 116023, China

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Abstract: A new approach to parameter estimation in groundwater hydrology is developed using hybrid ant colony system with simulated annealing. Based on the information from the observed water heads and calculated water heads, an objective function for inverse problem is proposed. The inverse problem of parameter identification is formulated as an optimization problem. Simulated annealing has the ability of probabilistic hill-climbing and is combined with ant colony system to produce an adaptive algorithm. A hybrid ant colony optimization is presented to identify the transmissivity and storage coefficient for a two-dimensional, unsteady state groundwater flow model. The ill-posedness of the inverse problem as characterized by instability and non-uniqueness is overcome by using computational intelligence. As compared with the gradient-based optimization methods, hybrid ant colony system is a global search algorithm which can find parameter set in a stable manner. A numerical example is used to demonstrate the efficiency of hybrid ant colony system.

Keywords: *Ant colony system; parameter identification; inverse problem; simulated annealing.*

Mathematics Subject Classification (2000): 65N21.

* Corresponding author: lishouju@dlut.edu.cn

1 Introduction

Parameter identification, or model calibration, is a critical step in the application of mathematical models in hydrologic sciences. Unfortunately, parameter identification is an inherently difficult process and, as an inverse problem, it is plagued by the well-documented problems of nonumiquenes, nonidentificability and instability [1]. Numerous optimization techniques have been used to solve groundwater remediation design and parameter identification problems. In a parameter identification problem, the objective function can be the weighted difference between the observed and calculated values at certain observation points in the aquifer. The identified parameters can be hydraulic conductivity or other aquifer parameters, such as storage coefficient [2]. In recent years, global optimization methods are being increasingly used to solve groundwater remediation design and parameter identification problems. These methods include simulated annealing, genetic algorithm, tabu search and ant colony system. As compared with gradient based local search methods, global optimization methods do not require the objective function to be continuous, convex, or differentiable. They have also shown other attractive features such as robustness, ease of implementation, and the ability to solve many types of highly complex, nonlinear problems. One common drawback of these global optimization methods is that many objective function evaluations are typically required to obtain optimal or near-optimal solutions [3]. Karimi investigated the problem of robust dynamic parameter-dependent output feedback (RDP-DOF) stabilization under H1 performance index for a class of linear time invariant parameter-dependent (LTIPD) systems with multi-time delays in the state vector and in the presence of norm-bounded non-linear uncertainties [4]. Li proposed a new interpretation to solve the inverse heat conduction problem using hybrid genetic algorithm. In order to identify parameters of non-linear heat transfer efficiently and in a robust manner, the hybrid genetic algorithm, which combines genetic algorithm with simulated annealing and the elitist strategy, is presented for the identification of the material thermal parameters [5]. Lou studied the robust stability of nonlinear uncertain neural networks with constant or time-varying delays. An approach combining the Lyapunov-Krasovskii functional with the linear matrix inequality is taken to study the problems [6]. Yu proposed an online learning algorithm for feedforward neural networks (FNN) based on the optimized learning rate and adaptive forgetting factor for online financial time series prediction [7]. The ant colony system is a kind of natural algorithm inspired by behavior or processes presented in nature. Ant colony system has been widely used in the traveling salesman problem, job-shop scheduling problem and quadratic assignment problem. When compared with traditional first-order methods, the ant colony system is recognized to have a better capability to find the global optimum solution. The objective of this paper is to present a new method based on hybrid ant colony system for obtaining the parameters of a linear groundwater flow model.

2 Calculation of Groundwater Flow Models

The partial differential equation for groundwater flow, assuming constant fluid density and viscosity, can be expressed as follows

$$\nabla \cdot (K \nabla h) + w = S_s \partial h / \partial t, \tag{1}$$

where h is the hydraulic head, K is the hydraulic conductivity tensor, w is the fluid sink/source term, and S_s is the specific storage coefficient of the aquifer. The first kind boundary condition is expressed as follows:

$$h(x,y,z)|_{\Gamma_1} = h_0(x,y,z), \tag{2}$$

where $h_0(x,y,z)$ is the already known head. The second kind boundary condition is written as follows:

$$Q_0(x,y,z)|_{\Gamma_2} = k_x\frac{\partial h}{\partial x}l_x + k_y\frac{\partial h}{\partial y}l_y + k_z\frac{\partial h}{\partial z}l_z, \tag{3}$$

where $Q_0(x,y,z)$ is the drainage already known, l_x, l_y and l_z are the direction cosines of the exterior normal of the boundary in x, y and z direction, respectively. And k_x, k_y and k_z are permeability coefficient in x, y and z direction, respectively. When the boundary condition and the predicted permeability coefficient are determined, the finite element equation is adopted to compute the distribution of the water head and the drainage in the whole seepage field, which provides modal data to the analysis of the inversion problem of the permeability coefficient. The numerical methods are based on spatial and temporal discretization which divides the continuous space and time domains into a network of discrete nodal points and a series of finite time intervals. When the various aquifer parameters are known, the hydraulic heads at any nodal points and time intervals can be obtained using finite-element code.

3 Classical Ant Colony System

Ant colonies have always fascinated human beings. Social insects, such as ants, bees, termites and wasps, often exhibit a collective problem-solving ability [8].The ant colony system is first applied to the traveling salesman problem. In Ant System, the traveling salesman problem is expressed as a graph (N, E), where N is the set of towns and E is the set of edges between towns. The objective of the traveling salesman problem is to find the minimal length closed tour that visits each town once. Each ant is a simple agent to fulfill the task. It obeys the following rules: 1) It chooses the next town with a probability which is a function of the town distance and of the amount of trail present on the connecting edge; 2) before a tour is completed, it can not choose the already visited towns; 3) when it completes a tour, it lays a substance called trail on each edge (i,j) visited; 4) it lives in an environment where time is discrete.

It must choose the next town at time t, and be there at time $t+1$. Let m, n be the total number of ants and towns. An iteration of the Ant system is called, as the m ants all carry their next moves during time interval $(t,\ t+1)$. The n iterations constitute a cycle. In one cycle, each ant has completed a tour. Let $\tau_{ij}(t)$ denote the intensity of trail on edge (i,j). After a cycle, the trail intensity is updated as [9]:

$$\tau_{ij}(t+1) = \rho\tau_{ij}(t) + \Delta\tau_{ij}, \tag{4}$$

where ρ is a coefficient, and $1-\rho$ represents the evaporation of trail between times t and $t+1$

$$\Delta\tau_{ij} = \sum_{k=1}^{Z}\Delta\tau_{ij}^{k}, \tag{5}$$

where $\Delta\tau_{ij}$ is the quantity per unit of length of trail substance placed on path(i,j) by the k-th ant between times t and $t+1$ [10]

$$\Delta\tau_{ij}^{k} = \begin{cases} Q/J_k, & path(i,j) \quad is \quad selected \\ 0, & otherwise \end{cases}, \tag{6}$$

where Q is a constant related to the quantity of trail laid by ants. Ants build solutions using a probabilistic transition rule. The probability $p_{ij}^k(t)$ with which ant k in town i at iteration t chooses the next town j to move to is a function of the heuristic function of the desirability η_{ij} and the artificial pheromone trail $\tau_{ij}(t)$:

$$p_{ij}^k(t) = \frac{[\tau_{ij}]^\alpha [\eta_{ij}]^\beta}{\sum\limits_{l=1}^{L} [\tau_{ij}]^\alpha [\eta_{ij}]^\beta}, \tag{7}$$

where α, β are adjustable constants, which can weigh the relative importance of pheromone trail and of objective function. A reasonable heuristic function is written as follows:

$$\eta_{ij} = \frac{1}{J_k}, \tag{8}$$

where J_k is the objective function of k-th ant path. Ant colony system could not perform well without pheromone evaporation. From Eq. (7), it is obvious that the transition probability is proportional to the visibility and the trail intensity at time t. The visibility shows that the closer towns have a higher probability of being chosen. The mechanism behind this is a greedy constructive heuristic. While the trail intensity shows that the more trail on edge (i, j), the more attractive it is. The process can be characterized by a positive feedback loop, in which an ant chooses a path thus reinforces it. In order to constrain the ants not to visit a previous visited town, a data structure called the tabu list is associated with each ant. All the visited towns are saved in it. When an ant finishes a cycle, the tabu list is then emptied and the ant is free again to choose. Let tabu $_k$ denote the tabu list of the k-th ant.

4 Parameter Estimation Approach in Groundwater Hydrology Using Hybrid Ant Colony System

4.1 Solution definition of inverse problem and its ill-posedness

The parameter identification problem can be formulated to find the model parameters by adjusting m until the measured data match the corresponding data computed from the parameter set in a least-squares fashion. The objective function is defined as follows [11]

$$J(m) = [h_m - h_c(m)]^T w [h_m - h_c(m)], \tag{9}$$

where h_m is the measured displacement vector; h_c is the computing displacement vector, which is related to the identified parameter vector m. w is weighting matrix in order to take into account the different observed equipments for the water head measurements. This objective function clearly depends on the measured data and the parameters of model.

Figure 4.1 shows the groundwater flow model, in which the four observing points for water heads are set in order to get measurement data and to identify the aquifer parameters. The objective function can become complex as shown in Figure 4.1, such as non-convex, or even multi-modal if errors contained in the model equation or /and errors in the measurement data are large. The multi local minima can be found from Figure 4.1. In such a case, the solution may vibrate or diverge when conventional gradient-based optimization methods are used, which gives rise to the necessary for a robust optimization method such that a stable convergence is always achieved.

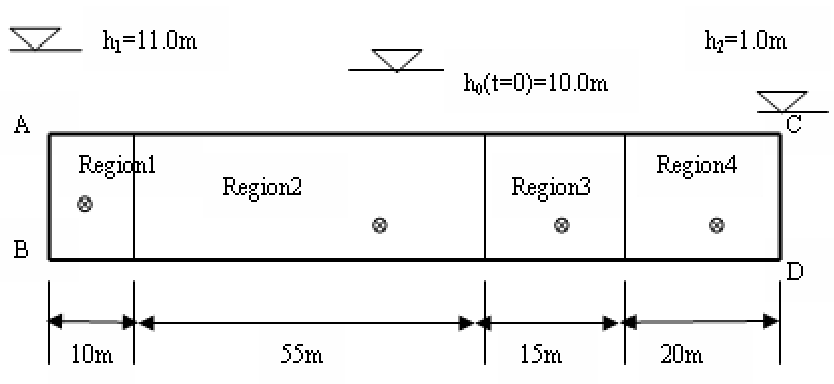

Figure 4.1: Configuration of the two-dimensional groundwater flow model.

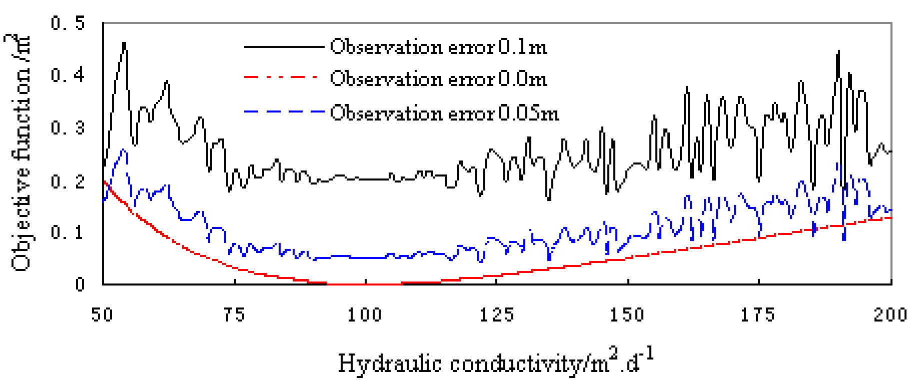

Figure 4.2: Configuration of the two-dimensional groundwater flow model.

The solution of the inverse problem consists in obtaining a minimum of an objective function which is defined taking into account the mathematical structure of the material model and asset of experimental data. This generally results in a non-linear programming constrained problem of the form [12]:

$$\min\{J(m, h_m)\ ,\quad m \in R^{\ P},\, h_m \in R^M;\, g^j < 0\},\tag{10}$$

where m represents the variable vector, which belongs to the space of admissible parameters R^P, h_m the vector of measured data, which belongs to the space R^M. g^j are inequality constraints, which define the feasible domain S:

$$S = \{m \in R^P, g^j < 0\}.\tag{11}$$

The constraints can represent physical links between the primary physical variables and the model parameters, information concerning the values of parameters and conditions to guarantee that all mathematical functions involved can be defined and calculated. In the optimization process, the difference between the experimental result and the theoretical prediction is measured by a norm value, here referred to as the individual norm. The individual norms of the tests form an objective function $F(x)$ which then gives a scalar measure of the error between the experimental observations and the model predictions. From mathematical point of view, the optimization problem involves the minimization of the objective function [13]:

$$J(m) \to \min,\tag{12}$$

where m is a vector containing the optimization variables (here model parameters) with the bound constraints:

$$m_l < m < m_u,\tag{13}$$

where m_l and m_u are the lower and upper bounds of m respectively.

4.2 Simulated annealing algorithm for neighborhood search

Simulated annealing is another important algorithm which is powerful in optimization and high-order problems. It uses random processes to help guide the form of its search for minimal energy states. Simulated annealing is a generalization of a Monte Carlo method for examining the equations of state and frozen states of n-body systems. The concept is based on the manner in which liquids freeze or metals recrystalize in the process of annealing [14]. In an annealing process a melt, initially at high temperature and disordered, is slowly cooled so that the system at any time is approximately in thermodynamic equilibrium. As cooling proceeds, the system becomes more ordered and approaches a "frozen" ground state at T=0. Hence the process can be thought of as an adiabatic approach to the lowest energy state. If the initial temperature of the system is too low or cooling is done insufficiently slowly the system may become quenched forming defects or freezing out in meta-stable states, that is, trapped in a local minimum energy state. Simulated annealing is a very general optimization method which stochastically simulates the slow cooling of a physical system. The idea is that there is a objective function F, which associates an objective function with a state of the system, a temperature T, and various ways to change the state of the system. The algorithm works by iteratively proposing change and either accepting or rejecting each change. Having proposed a change we may evaluate the change δ in F. The proposed change may be accepted or rejected by the Metropolis criterion; if the objective function decreases ($\delta J < 0$), the change is accepted unconditionally; otherwise it is accepted but only with probability $exp(-\delta J/T)$. For given old solution, a new solution can be created as follows [15]:

$$m_{new} = m_{old} + \Delta m, \tag{14}$$

where Δm is a random perturbation of solution. The accepted probability of the new solution, p_{new}, will be expressed:

$$p_{new} = \begin{cases} 1, & J_{old} \geq J_{new} \\ \exp[-\delta J/T_k], & J_{old} < J_{new} \end{cases}, \tag{15}$$

where $\delta J = J_{new} - J_{old}$. Three parameters essential for implementation of the simulated annealing algorithm are as follows: 1) initial value of the control parameter T_i, 2) the number of perturbations generated at each T, and 3) the decrement of the control parameter T. These parameters affect the speed of the algorithm and the quality of the final solution. A simple approach is to choose a value for T_i, that allows a large percentage of non-improving solutions to be accepted [14]. The number of solutions generated at each T is selected to allow equilibrium to take place before decreasing T. The decrement of T is chosen such that it allows only small changes in the value of T. The equation used for decreasing T is expressed as follows:

$$T_{k+1} = \xi T_k, \tag{16}$$

where $\xi = 0.9$ is a typical selection. A crucial requirement for the proposed changes is reachability or ergodicity that there be a sufficient variety of possible changes that one can always find a sequence of changes so that any system state may be reached from any other. When the temperature is zero, changes are accepted only if F decreases, an algorithm also known as hill-climbing, or more generally, the greedy algorithm. The initial temperature can be determined as [16]

$$T_i = -\frac{1}{\ln p_i}, \tag{17}$$

where p_i is the desired initial acceptable probability. It is usually between 0.7 and 0.9. similarly, the final temperature can be determined as

$$T_f = -\frac{1}{\ln p_f}, \tag{18}$$

where p_f is the desired final acceptable probability. It is usually very close to zero. The system soon reaches a state in which none of the proposed changes can decrease the objective function, but this is usually a poor optimum. In real life, we might be trying to achieve the highest point of a mountain range by simply walking upwards; we soon arrive at the peak of a small foothill and can go no further. On the contrary, if the temperature is very large, all changes are accepted, and we simply move at random ignoring the cost function. Because of the reachability property of the set of changes, we explore all states of the system, including the global optimum. The system evolves until a stop criterion is reached.

Very fast simulated annealing scheme proposed by Ingber is applied to produce new solution [17]

$$\Delta m_{\mathrm{i}} = \eta_{\mathrm{i}}(m_{\mathrm{i}max} - m_{\mathrm{i}min}), \tag{19}$$

$$\eta_i = sign(u_{\mathrm{i}} - 0.5)T_{\mathrm{i}}[(1 + \frac{1}{T_{\mathrm{i}}})^{|2u_i - 1|} - 1], \tag{20}$$

where m_{max} and m_{min} represent up and down bounds for parameters respectively; u_i is a random value in [-1,1) domain, the value of η_i is just located in [-1,1). The whole process for parameter identification using simulated annealing is shown as follows: Step 1: Initial parameters(initial T and temperature descent rate α) are fixed; Step 2: Initial solution is generated and the corresponding J is calculated; Step 3: The system solution is updated according to the mechanism designed; Step 4: Parameter T is modified according to the descent rate established; Step 5: One comes back to the step 3 to calculate the next solution from the current one up to T or ε reaches a value fixed beforehand; Step 6: The best solution visited is written as last solution of inverse problem.

4.3 Hybrid ant colony system for parameter identification

Ant colony system is global search techniques for optimization. However, it is poor at hill-climbing. Simulated annealing has the ability of probabilistic hill-climbing. Therefore, the two techniques are combined here to produce a new algorithm that has the merits of both ant colony system and simulated annealing, by introducing a local search. A new hybridization of ant colony system with simulated annealing is proposed. The main concept of inverse problem of parameter identification with the hybrid ant colony system can be summarized in the following steps: Step 1: depict each unknown parameter by an interval based on available prior information; Step 2: discretize each interval into a number of strata, let the middle of each stratum represent that stratum; Step 3: run the simulation model of choice for all, or a randomly selected subset of all the possible parameter combinations; Step 4: evaluate each stratum on the basis of the smallest value of the objective function in such a way that small values of the objective function receive higher scores; Step 5: Produce new individual based on SA neighborhood structures; Step 6: Accept new individual based on SA accepted probability; Step 7: on the basis of the value of the objective function, place a certain amount of trail(pheromone in the case of real ants) on each stratum visited along its pathway; Step 8: Decrease temperature of SA according to decreasing scheme; Step 9: repeat step 4 to step 8 until some convergence criterion is satisfied.

In order to study the performance of the proposed strategy, extensive numerical experimentations have been performed with Schaffer's function. The Schaffer's function is expressed as follows

$$f(x,y) = 0.5 - \frac{\sin^2(x^2+y^2)^{0.5} - 0.5}{(1+0.001(x^2+y^2))^2}, \qquad -4 < x, y < 4. \tag{21}$$

Figure 4.3 shows the fundamental structure of hybrid ant colony system with simulated annealing.

The Schaffer's function shape is shown in Figure 4.3 and searching values with HANC is listed in Table 4.1. The convergence processes of objective function with different algorithms are shown in Figure 4.3.

To test the applicability and efficiency of the hybrid ant colony system, a two dimensional flow problem is considered, as shown in Figure 4.1. The aquifer is bound by two constant-head boundaries with the initial heads both at 11m. at $t > 0$, the head at right boundary is instantaneously lowered to 10m. the transient head distribution is simulated

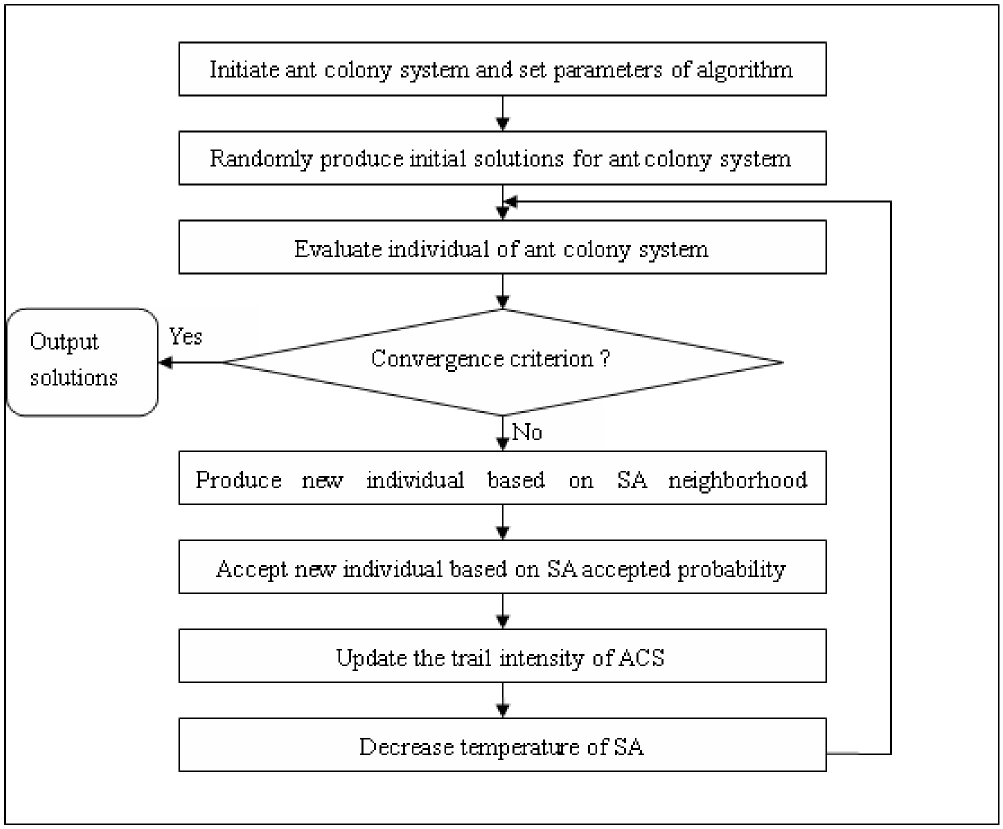

Figure 4.3: Fundamental structure of hybrid ant colony system with simulated annealing.

Theoretical values			Searching values with HANC		
x	y	f	x	y	f
0.00	0.00	1.00	0.00	-0.00	0.9999

Table 4.1: Computational values of Schaffer's function with hybrid ant colony system.

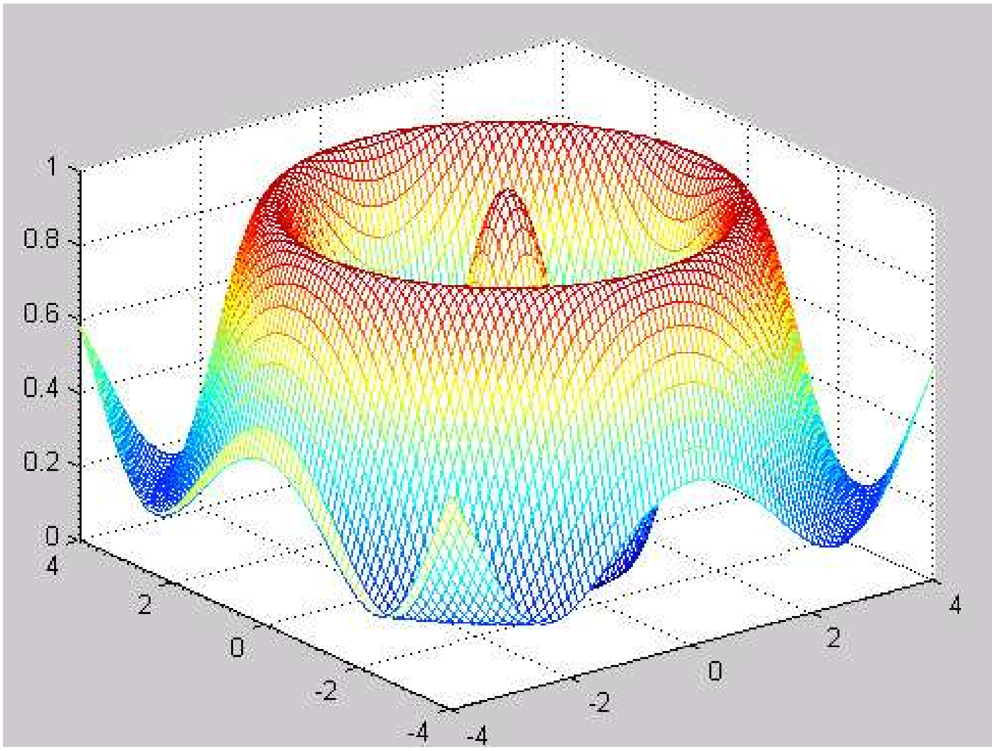

Figure 4.4: Schaffer's function shape.

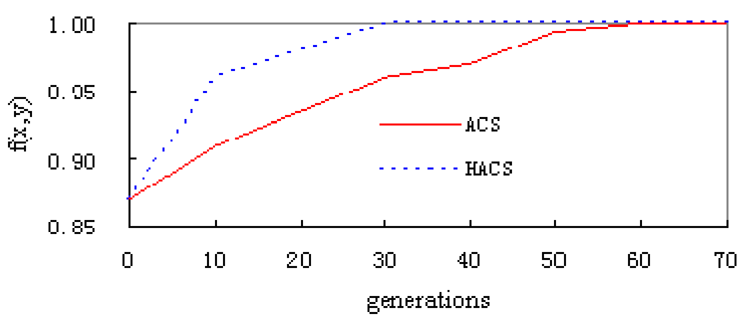

Figure 4.5: Search processes of Schaffer's function.

using finite element method with a time step of 0.1 day. Hydraulic conductivity and specific storage coefficient are listed in Table 4.2. Table 4.3 records measured water head data at different points at different times.

Parameters	k_1 / m^2·d^{-1}	k_2 / m^2·d^{-1}	k_3 / m^2·d^{-1}	k_4 / m^2·d^{-1}	S_s
Values	2000.0	100.0	10.0	1000.0	0.02

Table 4.2: Hydraulic conductivity and specific storage coefficient.

Observation time/d	Observation point 1# /m	Observation point 2#/m	Observation point 3#/m	Observation point 4#/m
0.1	10.687	9.9798	9.1706	3.640
0.2	10.728	9.8074	8.396	3.374
0.3	10.697	9.5376	7.756	3.121
0.4	10.635	9.198	7.028	1.584
0.5	10.560	8.835	6.398	1.423

Table 4.3: Measured water head data at different points at different times.

According to the flow mathematical model with finite element method and measured water-head data, the aquifer parameters are identified with hybrid ant colony system with simulated annealing. Figure 4.3 shows the influence of initial temperature of simulated annealing algorithm on convergence process.

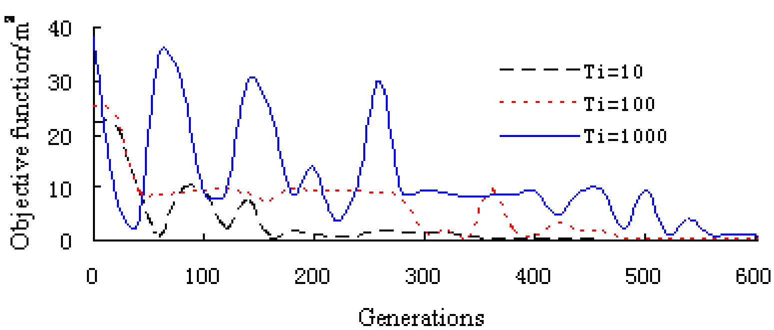

Figure 4.6: Influence of initial temperatures on convergence process.

In order to simulate observation errors, the measured water heads can be simulated by adding a random error to the theoretical values

$$h_m^* = h_m + sign(R - 0.5) \times \Delta h, \qquad (22)$$

where h_m^* are the measured data with observation errors, h_m are the measured data without observation errors. $Sign$ is the sign function, and R is a random variable in the interval [0,1], Δh is an observation error. Comparison of identified hydraulic conductivity and specific storage coefficient with theoretical values is listed in Table 4.4. Figure 4.3 shows the convergence process of objective function with classical ant colony system and hybrid ant colony system.

Model parameters	$k_1/m^2 \cdot d^{-1}$	$k_2/m^2 \cdot d^{-1}$	$k_3/m^2 \cdot d^{-1}$	$k_4/m^2 \cdot d^{-1}$	S_s
Theoretical values	2000.0	100.0	10.0	1000.0	0.02
Identified values by HACS	2010.3	99.7	10.2	1009.0	0.018
Identified values by ACS	2022.7	110.2	9.95	1012.5	0.025

Table 4.4: Comparison of identified hydraulic conductivity and specific storage coefficient with theoretical values.

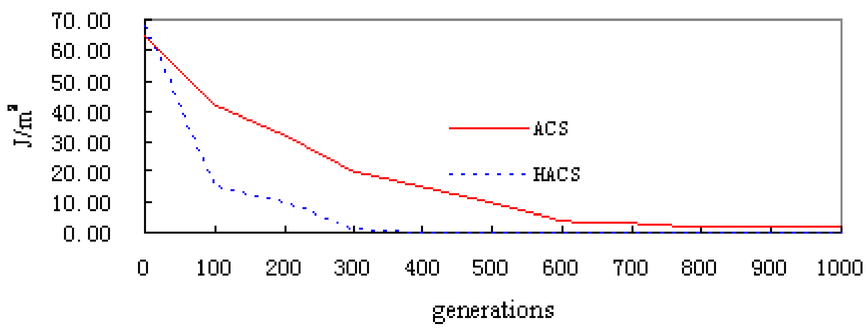

Figure 4.7: Convergence process of objective function with different searching method.

5 Practical Applications of Inversion Algorithm

Baishan Hydropower Station, as shown in Figure 5.1, is located in the Second Songhuajiang River in Jilin province, China. It consists of a 149.5-meter-high concrete heavy-pressure dam, a weir with four 12×13 meter tunnels on top of the 404-meter-high spillway dam, three 6×7 meter tunnels for discharging water are 350 meters high, an underground powerhouse with an installed generating capacity of 900,000 KW and another powerhouse on the surface with an installed generating capacity of 600,000 KW. The dam is 423.5 meters high and the reservoir has a storage capacity of 6.812 billion cubic meters. Its highest normal storage water level is 413 meters. The capacity for water control storage is 3.54 billion cubic meters while the flood control storage capacity is 950 million cubic meters. Cross-section of Baishan dam at block 18 is shown in Figure 5.2. Figure 5.3 shows the disposition of observation holes for dam uplift pressure at block 18.

In order to identify the permeability coefficients of rock foundation, the three-dimensional finite element model for seepage calculation is carried out. The seepage fields of the dam and its rock foundation at different load cases are computed. According to the prior information of pumping water test in field, the domains of identification parameters are determined. The training sample pairs are got basing on finite element

Figure 5.1: Baishan Hydropower Station.

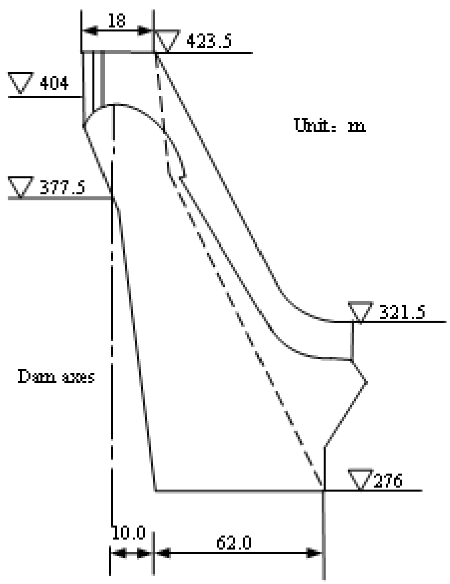

Figure 5.2: Cross-section of Baishan dam at block 18.

Measuring date	Upstream water elevation	Downstream water elevation	water head h_1	water head h_2	water head h_3	water headh_4
19980910	413.00	290.80	291.82	283.59	284.74	281.72

Table 5.1: Measured data of water heads in the observation holes.

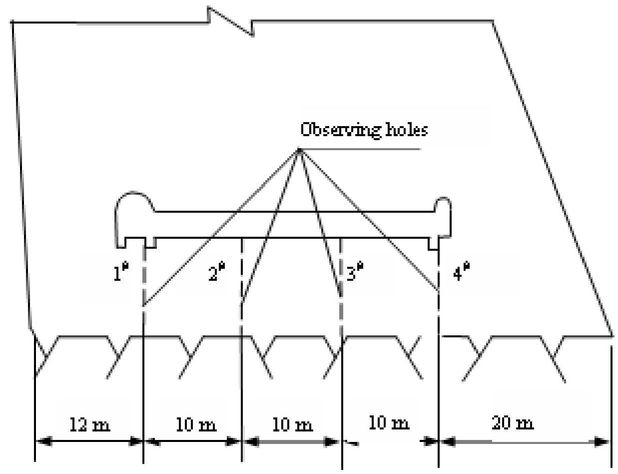

Figure 5.3: Disposition of observation holes for dam uplift pressure at block 18.

Rock foundation (I) $k_1/10^{-9}m \cdot s^{-1}$	Concrete certain $k_2/10^{-9}m \cdot s^{-1}$	Rock foundation (II) $k_3/10^{-9}m \cdot s^{-1}$
44.05	6.16	52.80

Table 5.2: Identification results of permeability coefficients.

Measuring date	No.1 measured point	No.2 measured point	No.3 measured point	No.4 measured point
19971015	291.82/291.78	283.54/283.49	284.23/283.19	281.72/281.13
19980108	291.31/291.29	283.59/283.44	282.19/283.15	281.82/282.14
19980901	291.82/291.78	283.59/283.49	284.74/283.19	281.72/282.12
19981012	291.82/291.78	283.49/283.49	284.74/283.19	281.72/282.12

Table 5.3: Comparison between measured and forecasted water heads.
Note: measured water heads/computed water heads.

analysis. The rock foundation is divided into 3 sub-regions, rock base before concrete certain, concrete certain and rock base after concrete certain.

The measured water heads in four holes are recorded in Table 5.1. Based on the measured water heads and finite element model for dam seepage calculation, the permissibility coefficients are identified and listed in Table 5.2. Table 5.3 shows the comparison between measured and forecasted water heads with finite element method according to estimated permissibility coefficients.

6 Conclusion

Hybrid ant colony system for solving the parameter identification problem is proposed. The three characteristics of the ant colony system, such as positive feedback process, greedy constructive heuristic and distributed computation, work together to find the solution to the inverse problems fast and efficiently. However, classical ant colony system is poor at hill-climbing. Simulated annealing has the ability of probabilistic hill-climbing. Therefore, the two techniques are combined here to produce a new algorithm that has the merits of both ant colony system and simulated annealing, by introducing a local search. A new hybridization of ant colony system with simulated annealing is proposed.. Modern heuristic search techniques, such as genetic algorithm, simulated annealing and ant colony system, are well suited for solving the parameter identification problem in groundwater flow model. The gradient based methods are not applicable for this type of inverse problem because of the difficulty in evaluating the function derivatives and the presence of many local minimum points in the objective function. One of the advantages of ant colony system over other optimization methods is that it is easy to implement complex inverse problem.

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A Survey of the Dynamics and Control of Aircraft During Aerial Refueling

W. Mao and F.O. Eke *

Department of Mechanical and Aeronautical Engineering
University of California Davis
Davis, CA 95616, USA

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Abstract: Recent heightened interest in autonomous refueling of unmanned aerial vehicles has stimulated research activity in the area of aerial refueling in general. Aircraft aerial refueling research can be divided into four general areas: influence of tanker aircraft wake turbulence on the receiver aircraft, the dynamics of the drogue and hose, automatic flight control system design for aerial refueling, experiments and flight tests related to the practical implementation of autonomous aerial refueling system. This survey summarizes research activities as well as the current state of knowledge in these areas.

Keywords: *Aerial refueling; variable mass system; aircraft dynamics.*

Mathematics Subject Classification (2000): 93C20, 93C35, 93C85.

1 Introduction

Aerial refueling is the practice of transferring fuel from one aircraft to another during flight. It allows the receiving aircraft to remain airborne longer, and to take off with a greater payload. Aerial refueling operation with manned aircraft has been implemented by many countries since after the Second World War. In-flight refueling was first proposed in 1917 by Alexander P. de Seversky, who was then a pilot in the Russian Navy. The motive was to increase the range of combat aircraft. De Seversky soon emigrated to the United States and became an engineer in the War Department. He initiated work on Aerial Refueling in the United States. Although experiments in aerial refueling started as

* Corresponding author: foeke@ucdavis.edu

early as the 1920s, hardly any analytical study in this area of research was conducted until the 1980s. The purpose of this paper is to give an overview of research work conducted to date on in-flight refueling of flight vehicles.

Early research work on aerial refueling concentrated on the aerodynamics aspect of aerial refueling, mainly the influence of the tanker wake turbulence on the stability and control of the receiver aircraft. In recent years, more and more Unmanned Air Vehicles (UAVs) are used in military operations. As UAVs are expected to perform functions similar to those of manned aircraft, UAVs are also expected to have an aerial refueling capability. This expectation has motivated much of recent research work on aerial refueling. The ultimate goal of aerial refueling research is to develop reliable automatic flight control systems which can guide UAVs or manned aircraft through aerial refueling operation.

2 Aerial Refueling Methods

There are two methods commonly used for aerial refueling: Probe and Drogue Refueling (PDR) and Boom and Receptacle Refueling (BRR). In the probe and drogue method, the tanker aircraft releases a long flexible hose that trails behind and below the plane. At the end of the hose is a cone-shaped component known as a drogue or basket. A plane that needs to refuel extends a device called a probe, which is a rigid, sometimes jointed, arm placed usually on one side of the airplane. As the tanker flies straight and level with no control on the drogue, the pilot of the receiving aircraft flies his airplane behind and below the tanker aircraft and in such a way that the probe mounted on the receiver aircraft links up with the drogue from the tanker. Once the connection is made, a valve in the drogue opens to allow fuel to be pumped through, and the two aircrafts fly in formation until the fuel transfer is complete. The receiver aircraft then decelerates hard enough to pull the probe out of the basket. PDR is the standard aerial refueling procedure for the US Navy (USN), North Atlantic Treaty Organization (NATO) nations, Russia, and China.

In the boom and receptacle refueling method, the boom is a long, rigid, hollow shaft, usually fitted to the rear of the tanker aircraft. It normally has a telescoping extension, a valve at the end to keep fuel in and permit it to flow when necessary, and small wings to enable it to be "flown" into a receptacle of the aircraft to be refueled. The plane that is to receive fuel is equipped with a receiver socket fitted onto the top of the aircraft, on its center line and usually either behind or close to the front of the cockpit. The receiver socket is a round opening which connects to the fuel tanks, with a valve to keep the fuel in when the plane is not being refueled, and dust and debris out. The boom has a nozzle which fits into this opening. During refueling operations, the tanker aircraft flies in a straight and level attitude at constant speed, while the receiver takes a standard position behind and below the tanker. As the receiver pilot flies in formation with the tanker, the boom operator in the tanker's tail uses a joystick to move the boom and extend the telescoping component to connect the boom's nozzle to the receiver. When an electrical signal is passed between the boom and receiver, the valves in both the boom and the receiver are opened. Pumps on the tanker drive fuel through the boom's shaft and into the receiver. When refueling is complete, the valves are closed and the boom is retracted. BRR is the preferred refueling method for the US Air Force (USAF). In addition to the US Air Force, BRR is also used by the Netherlands, Israel , Turkey, and Iran.

Compared to BRR, PDR is simpler and more flexible in its implementation. Many types of aircraft can be modified to carry drogue systems. The PDR system allows multiple aircraft being refueled simultaneously, and requires no extra boom operator. However, PDR relies on the receiver aircraft to make the refueling connection, which can be a demanding task especially for a fatigued pilot or during night/bad weather operations. PDR also provides a lower fuel transfer rate in general. In BRR, the receiver pilot's workload is slightly lower, and BRR also provides higher fuel transfer rate. However, the tanker can only service one receiving aircraft at a time. The space and weight associated with the boom assembly puts a restriction on the types of aircraft that can be equipped with this system.

An aerial refueling procedure can be divided into three phases: the pre-refueling or approach phase, the refueling phase, and the separation phase. In the approach phase, the receiver aircraft approaches the tanker aircraft from below and behind and gets connected with the tanker. During the refueling phase, fuel is pumped from the tanker aircraft into the receiver aircraft. The receiver aircraft tries to hold a stationary position relative to the tanker aircraft to maintain the connection between drogue and probe, or boom and receptacle. This phase can also be called the station keeping phase. The separation phase begins as soon as fuel transfer ends. The receiver aircraft decelerates and becomes detached from the tanker aircraft.

Flying the receiver aircraft during aerial refueling, especially during the first two phases, is much more difficult than under normal flight condition because of tanker wake turbulence. Furthermore, as the receiver aircraft approaches the tanker aircraft in PDR, the relative position of the hose and drogue fluctuates due to wind gusts and turbulence. It is not a trivial task to make the connection between the drogue and the probe. For a manned aircraft, such difficulties can be overcome by a pilot's agility. For UAVs, these difficulties impose challenges that must be resolved through automatic flight control system design.

3 Aerodynamic Effects on the Receiver Aircraft

Although practical attempts at aerial refueling started in the 1920s, there was little theoretical research work related to aerial refueling until the 1980s when the needs of simulation software for pilot training made it necessary. One of the first questions to be addressed by researchers is whether the aerodynamic impact of the tanker wake turbulence on the receiver aircraft is significant.

Bloy et al [6] studied the lateral dynamic stability and control of a large receiver aircraft during aerial refueling in 1986. The probe and drogue refueling approach was assumed in the study. The receiver aircraft was taken to be at a typical refueling location approximately two wing spans directly behind and a quarter wing span below the tanker aircraft. In the study, a simple horseshoe vortex model was assumed for the tanker vortex field. Due to the effect of the tanker vortex field, two additional derivatives were found to be required for studying the dynamics of the receiver. These derivatives are the rolling moment due to bank and the rolling moment due to sideways displacement. It was found that these derivatives are both negative, which means that the receiver aircraft is statically stable with respect to lateral displacement and bank attitude. To study the lateral dynamic stability of the receiver aircraft, the linearized lateral equations of motion for initially steady, straight, horizontal flight were used. These equations were then written in the generalized Eigenproblem form $Ax = \lambda Bx$, and dynamic modes were calculated. The receiver aircraft was found to exhibit divergent oscillations involving mainly bank and sideways displacements under the influence of the tanker vortex field.

Bloy et al [3] extended their work to the study of the longitudinal dynamic stability and control of a large receiver aircraft during aerial refueling in 1987. The assumptions are the same as those in the previous study. For the dynamics of longitudinal motion, the two most important additional aerodynamic derivatives were the normal force due to vertical displacement and the pitching moment due to vertical displacement. These two derivatives were found to be negative, which means that the receiver is statically stable with respect to steady state. The linearized longitudinal equations of motion were derived, and were used to show that the receiver aircraft exhibits instability or near neutral stability in vertical displacement depending on the relative values of the mean span-wise downwash gradients at the receiver wing and tail-plane positions.

Bloy et al [8] further studied the longitudinal stability of receiver aircraft for different aerial refueling configurations. Two receivers, the VC10 and Hercules, were refueled from four tanker aircrafts, Victor, Hercules, VC10, and Tristar. It was found that the receiver aircraft longitudinal stability depends on several parameters: the vertical separation between the receiver and the tanker, tanker properties (wing-span and weight which all affect the vortex field produced), receiver properties (tail-plane height, mass, center of gravity position), flight speed, and attitude of aircraft. Among the parameters, the most important appeared to be the vertical separation between the receiver and the tanker. The vortex field at the receiver aircraft position varies with the separation distance. Thus, the drag and lift caused by the vortex field also change. Furthermore, the relative downwash at the wing and tail-plane of the receiver also changes, and this causes variation in the pitch moment on the receiver.

To verify their theoretical results, Bloy et al [9] conducted wind tunnel experiments with models of tanker and receiver aircraft. The tanker was modeled as a straight wing while the receiver aircraft was modeled as a wing and fin with tail-plane at low, medium, or high positions. Both the tanker model and the receiver model were put into a wind tunnel with relative position similar to that in aerial refueling. The aerodynamic forces and moments acting on the models were measured and compared with those obtained from theoretical computation. The experimental results were found to be in fairly good agreement with the theoretical predictions. Bloy et al further extended their wind tunnel experiments on tanker models with the use of a flapped wing [10] and a tapered wing [12, 7]. In these later cases, the theoretical and experimental trends were similar, although there were significant differences between theory and actual experimental data due presumably to wind tunnel boundary interference effects.

To consider the effect of different vortex models, a flat vortex sheet model and a more realistic roll-up vortex model were compared for the tapered tanker wing [11]. For the flat vortex model, the downwash over the central part of the receiver wing is less than that obtained from the wake model with roll-up. Towards the tip, the situation is reversed. The effect of wake roll-up on receiver rolling moment due to sideways displacement derivative was also calculated and compared with that for the flat vortex model. The wake roll-up model showed much higher rolling moment values from the comparison.

The studies of Bloy et al [6, 3, 8, 9, 10, 12, 7, 11] indicate that a tanker aircraft's wake turbulence has a significant impact on the receiver aircraft's dynamics. The subsequent question is how the influence of tanker wake turbulence should be accounted for in analytical studies and simulations. One approach is to consider the variable downwash and sidewash distribution of tanker wake turbulence on the lifting surfaces of the receiver aircraft, and determine the resultant aerodynamic forces and moments on the receiver aircraft using complicated computational fluid dynamics models. This method is called the exact model method. To ensure accuracy, computational predictions from such models are often verified with wind tunnel experiments. Blake et al [2] presented results from a wind tunnel testing of Innovative Control Effectors 101 (ICE101), a tailless aircraft configuration, behind a KC-135R tanker, and compared these results with predictions from a planar vortex lattice code. The aircraft models were 1/3 scale, and they were tested in a full scale wind tunnel. The KC-135R wake induced lift, drag, pitching moment, rolling moment, yawing moment and side force on ICE101 with different relative vertical and lateral positions were measured and compared with predictions from the planar vortex lattice model. Both the predictions and measured data show wake interference effects that vary significantly with relative lateral and vertical position, and weakly with relative longitudinal position. Results from wind tunnel tests and theoretical predictions were found to be in excellent agreement except for drag. The discrepancy in the drag results is believed to be due to the fact that viscous effects are ignored in the vortex lattice model.

Predicting aerodynamic forces and moments with an exact model method is computation intensive. A simpler method is to only consider the tanker wake conditions at the receiver's center of gravity, to assume linear distributions of downwash and sidewash on receiver aircraft lifting surfaces, and determine the resultant aerodynamic forces and moments. This method is called the single-point model method. In this method, calculations are greatly simplified. Bloy et al [4] proposed and validated the single-point model method for aerial refueling simulations. They found out that the single point method is adequate whenever the ratio of the wing span of the receiver aircraft to that of the tanker aircraft is much less than one. The method becomes less accurate as the wing span of the receiver aircraft is increased. The benefit of the single-point model is that it does not demand extensive computation, which is good for real time analysis.

Venkataramanan and Dogan [33, 16] developed another approximate method for the calculation of aerodynamic coupling between two aircraft flying close to each other, as is the case during aerial refueling. In the method proposed, the average wind velocity and the weighted average of wind velocity gradient on the surface of the trailing aircraft is taken to be the effective wind velocity and wind velocity gradient acting on the center of gravity of the aircraft. Svoboda and Ryan [29] developed an aerodynamic model of Boeing E-3A based on models of aerial refueling between two tankers, Boeing KC-135R and Douglas KC-10A, and five receivers, Lockheed C-141B, Lockheed C-5B, Douglas C-17A, Douglas KC-10A, and Boeing KC-135R. To establish the aerodynamic models of aerial refueling between these tankers and receivers, free air simulation and free air flight test were performed first and data collected. Then, flight test data were collected during aerial refueling. The effects of the tanker wake turbulence on the receiver were found out by comparing the difference between free air flight test data and aerial refueling flight test data. Simulation models for aerial refueling between these tankers and receivers were obtained from the comparison. As the configuration of Boeing E-3A is similar to Douglas DC-10A and Boeing KC-135R, the authors assumed that the aerial refueling model of Boeing E3A is an average of those for Douglas DC-10A and Boeing KC-135R.

In recent years, much research effort is focused on the development of autonomous aerial refueling of UAVs. Clearly, aerodynamic models for UAVs are required for this purpose. To protect proprietary data of different combat UAV manufacturers (Boeing and Northrop Grumman), an equivalent simulation model was developed [1] at the Air Force Research Laboratory (AFRL). This model allows simulation research and development to be conducted for automated aerial refueling of unmanned aerial vehicles. The

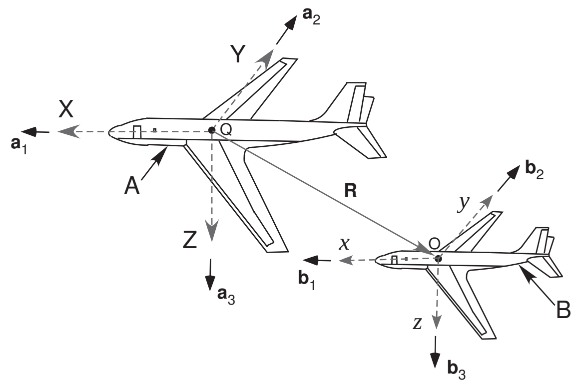

Figure 4.1: Aerial refueling.

model is developed based on ICE 101, whose configuration and aerodynamic data had been cleared for release to the public. Specifications independently provided by Boeing, Northrop Grumman, and a special design team at AFRL are combined to yield non-proprietary specification for the equivalent model. The physical, aerodynamic, and control characteristics of ICE 101 are then modified to satisfy the specifications. This modified model of ICE 101 is intended to be used for UAV automated aerial refueling research.

4 Effects of Mass Variation on the Dynamics of Receiver Aircraft

F. Eke and W. Mao [22] extend the study of dynamics of variable mass system [18, 17, 27, 34] to the dynamics of receiver aircraft during aerial refueling (Figure 4.1). In their study, the translational motion and rotational motion of the receiver aircraft are described by the following two equations.

$$\begin{aligned}\mathbf{F}_G + \mathbf{F}_E + \mathbf{F}_A \;=\; & (M_B + m_F)\Big[{}^N\mathbf{a}^Q + \ddot{\mathbf{R}} + 2\,{}^N\boldsymbol{\omega}^A \times \dot{\mathbf{R}} \\ & + {}^N\boldsymbol{\alpha}^A \times \mathbf{R} + {}^N\boldsymbol{\omega}^A \times \left({}^N\boldsymbol{\omega}^A \times \mathbf{R}\right)\Big] \\ & + m_F\left\{\left({}^N\boldsymbol{\alpha}^A + {}^A\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^A \times {}^A\boldsymbol{\omega}^B\right) \times \mathbf{r}_F^* \right. \\ & \left. + \left({}^N\boldsymbol{\omega}^A + {}^A\boldsymbol{\omega}^B\right) \times \left[\left({}^N\boldsymbol{\omega}^A + {}^A\boldsymbol{\omega}^B\right) \times \mathbf{r}_F^*\right]\right\} - \dot{m}_F \mathbf{v}_r\end{aligned} \tag{1}$$

and

$$\begin{aligned}\mathbf{M}_G + \mathbf{M}_E + \mathbf{M}_A \;=\; & m_F \mathbf{r}_F^* \times \Big[{}^N\mathbf{a}^Q + \ddot{\mathbf{R}} + 2\,{}^N\boldsymbol{\omega}^A \times \dot{\mathbf{R}} \\ & + {}^N\boldsymbol{\alpha}^A \times \mathbf{R} + {}^N\boldsymbol{\omega}^A \times \left({}^N\boldsymbol{\omega}^A \times \mathbf{R}\right)\Big] \\ & + (\mathbf{I}_B + \mathbf{I}_F) \cdot \left({}^N\boldsymbol{\alpha}^A + {}^A\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^A \times {}^A\boldsymbol{\omega}^B\right) \\ & + \left({}^N\boldsymbol{\omega}^A + {}^A\boldsymbol{\omega}^B\right) \times (\mathbf{I}_B + \mathbf{I}_F) \cdot \left({}^N\boldsymbol{\omega}^A + {}^A\boldsymbol{\omega}^B\right) \\ & + \left(\frac{{}^B d\mathbf{I}_F}{dt}\right) \cdot \left({}^N\boldsymbol{\omega}^A + {}^A\boldsymbol{\omega}^B\right) - \dot{m}_F \mathbf{r}_R \times \mathbf{v}_r \\ & - \dot{m}_F \mathbf{r}_R \times \left[\left({}^N\boldsymbol{\omega}^A + {}^A\boldsymbol{\omega}^B\right) \times \mathbf{r}_R\right],\end{aligned} \tag{2}$$

where $\mathbf{F}_G$, $\mathbf{F}_E$, $\mathbf{F}_A$ are gravity force, thrust, and aerodynamic force respectively; $\mathbf{M}_G$, $\mathbf{M}_E$, $\mathbf{M}_A$ are moment due to gravity force, engine's angular momentum, and aerodynamic moment respectively; M_B is the mass of receiver aircraft without fuel, m_F is the mass of fuel, $\mathbf{R}$ is the position vector of the receiver aircraft relative to the tanker aircraft, $\mathbf{r}_F^*$ is the position vector of fuel mass center in receiver's body frame, ${}^N\mathbf{a}^Q$ is the acceleration of the origin of the tanker's body frame, ${}^N\boldsymbol{\omega}^A$ is the angular velocity of the tanker aircraft relative to an inertia frame, ${}^N\boldsymbol{\alpha}^A$ is the angular acceleration of the tanker aircraft relative to an inertia frame, ${}^A\boldsymbol{\omega}^B$ is the angular velocity of the receiver aircraft relative to the tanker aircraft, ${}^A\boldsymbol{\alpha}^B$ is the angular acceleration of the receiver aircraft relative to the tanker aircraft, $\mathbf{v}_r$ is the fuel velocity at entry point relative to the receiver's body frame, $\mathbf{r}_R$ is the position vector of the fuel entry point relative to the receiver's body frame, $\mathbf{I}_B$ is the moment of inertia of the receiver aircraft without fuel, $\mathbf{I}_F$ is the moment of inertia of the fuel.

Results obtained from numerical simulations indicate that mass variation due to fuel transfer compounds the difficulties created by tanker wake turbulence. In order to keep the receiver aircraft at a fixed position relative to the tanker during aerial refueling, appreciable adjustments must be made to the receiver's angle of attack, throttle setting and elevator deflection. A larger refueling rate demands even larger adjustments. Changes in certain other parameters related to aerial refueling can also amplify the effects of mass variation on the receiver motion, or influence the system's dynamics in other ways.

5 Dynamics of Drogue and Hose

During air-to-air refueling with the hose and drogue approach, the hose and drogue trail down from a tanker aircraft. The hose and drogue are subject to aerodynamic forces, gravity, and tension. The vortex field of the tanker aircraft and the receiver aircraft nose, when the receiver is close enough, also have an impact on the hose and drogue. A good understanding of the dynamics of the hose and drogue makes the position of the drogue more predictable, which is important to the aerial refueling procedure. Bloy et al [5] proposed a static model of hose and drogue. They found that the motion of the drogue is well damped and the interference effect of the receiver nose on the drogue can be determined to satisfactory accuracy from the static analysis of the hose and drogue when the receiver aircraft approaches at a typical closure speed used in aerial refueling. In their study, finite element analysis was applied to the hose. A hose element is subject to the aerodynamic force on the element, gravity, and tension forces at two ends. Equilibrium equations can be written for the hose element. For all the hose elements, a set of equations can be obtained. Similarly, equilibrium equations can be written for the drogue. All these equations can then be solved iteratively to obtain the displacement of the end points of each element.

Researchers at NASA Dryden Flight Research center [31] used experimental methods to support the development of accurate aerodynamic models of the drogue and hose assembly to be used in refueling simulations. In flight tests, the thrust of the tanker aircraft was measured. The difference in the thrust measured when the drogue and hose were deployed and when they are stowed is attributable to the drag of the drogue and hose. Drag data were obtained at different airspeed and altitude. It was found that drag increases linearly with airspeed, but that there was no discernible altitude effect on drag. When the receiver airplane engaged the drogue, some of the aerodynamic load (drag) on the drogue and hose assembly was transferred from the tanker to the refueling probe of the receiver. There is also a linear trend in such drag relief as airspeed increases. Data obtained from flight tests were compared with wind tunnel results, and they were in good agreement.

Hansen et al [20] hypothesized that the position of the drogue relative to the tanker is a function of several independent variables and could be obtained by the superposition of the constituent effects. The variables considered are flight condition, drogue condition, hose weight effects, tanker effects, and receiver effects. Flight tests with two F/A-18 were designed to determine the change in drogue position as a function of individual influences. One of these two F/A-18 aircraft carried the aerial refueling store (ARS), and performed as the tanker, and the other F/A-18 acted as the receiver. Cameras were mounted on both aircrafts to monitor and measure the position and the movement of the drogue. Several effects on the hose and drogue were observed during the flight tests.

1. Free-Stream Drogue Position and Airspeed: It is observed that the drogue position is higher with faster tanker airspeed.

2. Free-Stream Drogue Position and Tanker Angle of Attack (AOA): As AOA increases, the drogue position becomes lower almost linearly.

3. Free-Stream Drogue Position and Turbulence: In light turbulence, the drogue did not stabilize; it randomly meandered in the horizontal and vertical directions by as much as a drogue diameter (approximately 0.6m).

4. Area of Influence (AOI): As the receiver approaches the drogue, the nose of the receiver has a measurable effect on the drogue position. The boundary of the AOI is defined by the locus of points at which the nose of the receiver has a minimum measurable effect on the drogue position.

Valuable data have been collected from flight tests [20, 31], and more flight tests have been planned for the complete modeling of the hose and drogue dynamics.

6 Automatic Flight Control System

Automatic flight control system (FCS) design for aerial refueling involves the selection of sensors for detecting the relative position of the tanker aircraft and the receiver aircraft, as well as the development of control laws to guide the aerial refueling process.

Valasek et al [32] proposed the use of an optical sensor fixed on the tanker aircraft to detect the position of the drogue. Several LED based beacons are attached to the drogue. Analysis of signals sent between the beacons and the optical sensor leads to the determination of the six degree-of-freedom sensor position and attitude data with respect to a reference frame fixed on the drogue.

The design and simulation of a controller for the docking procedure was also presented by Valasek et al [32]. Here, the drogue was assumed to be stationary. Tanker turbulence was treated as uncertain disturbance and was rejected by the control system. In the study, an Optimal Nonzero Set Point Controller is used in the flight control system of the receiver aircraft. The optimal controller developed by Valasek et al assumes that there are no exogenous inputs to the system. To improve the disturbance rejection properties of the controller to exogenous inputs, a low pass filter is used to pre-filter the control commands. Valasek et al simulated the system for the case of docking with a stationary drogue from an initial offset in three axes, with turbulence. They found that the system was able to effectively accomplish the docking task.

In the above control system design by Valasek et al [32], it was assumed that all the components of the state vector $\mathbf{x}$ are known. This may not be true in the real world. To estimate the state variables that are not provided by the optical sensor and also to filter out the process noises (gusts, tanker turbulence) and measurement noise for the sensors, Kimmett et al [21] improved the control system design by adding a variational Kalman Filter into the system. The controllers developed by Valasek et al [32] and Kimmett et al [21] are most suitable for tracking a relatively stationary drogue. Tandale et al [30] developed a Reference Observer Based Tracking Controller that does not require a model of the drogue or presumed knowledge of its position. A trajectory generation module is used to translate the relative drogue position measured by the sensor into a smooth reference trajectory, and an output injection observer is used to estimate the states to be tracked by the receiver aircraft.

Assume that the earth fixed inertial axis system (X_n, Y_n, Z_n) is oriented with the X_n axis pointing along the heading of both the tanker and receiver aircraft, and the Z_n axis points in the direction of gravity. The body axis (X_b, Y_b, Z_b)is attached to the receiver aircraft with the origin at its center of mass. Let (X_d, Y_d, Z_d) be the initial offset as measured along the inertial axis, between the mean position of the refueling drogue and the probe attached to the receiver aircraft. The drogue exhibits random oscillatory behavior in the plane parallel to the (Y_n, Z_n) plane and its mean position may be estimated by taking an average of the drogue position over a period of ten seconds prior to initiating the docking maneuver.

The reference trajectory is designed in two stages. In the first stage, the refueling probe on the receiver aircraft tries to line up behind the mean position of the drogue so that the initial offset (Y_d, Z_d) becomes zero. A smooth 5^{th}order polynomial trajectory is used to design the flight trajectory for the first stage. The parameters of this smooth spline are selected by imposing continuity, zero velocity, and zero acceleration at the initial and final times of the first stage. During the second stage, the probe follows the drogue positions along the Y_n and Z_n axis exactly. The reference trajectory is designed as a smooth reference trajectory between the mean drogue position and the current drogue position along the Y_n and Z_n axis. The reference trajectory which zeros the offset X_d is designed as a smooth 5^{th}order polynomial, but the initial and final times are the initial time of the first stage, and the final time of the second stage respectively.

To ensure that the reference trajectory is feasible and does not demand excessive rates in the states as well as the control, the time duration of the first and second stages are design parameters which must be judiciously selected as functions of the initial offset (X_d, Y_d, Z_d). The reference trajectory generated above is expressed in terms of the outputs δX, δY, and δZ respectively. The state feedback controller to be designed requires the knowledge of the full state vector for the reference trajectory. The purpose of the observer is to generate the reference states that the receiver aircraft should follow so that it can track the reference trajectory. The tracking performance of the Non-Zero Set Point Controller[32] was compared to the Reference Observer Based Tracking Controller (ROTC) [30]. ROTC shows less lag in the tracking performance and a 75% decrease in the tracking error.

Fravolini et al [19] proposed a fuzzy fusion strategy to combine information from GPS and machine vision system to determine the position of the drogue. They took d_G to be the distance between the receiver and the tanker as measured by GPS, and d_D the distance between the probe and drogue, as determined by a vision system. For large values of the distance between the tanker and receiver, the distance feedback measurement r_d

is provided by the GPS-based distance d_G. At intermediate distance, r_d is provided by a weighted combination of the GPS-based and the vision-based distance of the drogue. At small distance, r_d is the magnitude of the relative position vector of the drogue estimated by the machine vision system. When the refueling control system is activated at time t_0, the output of the fuzzy fusion system will be a large relative distance error $r_d(t_0) \neq 0$. To smooth the error signal to avoid actuator saturation or large accelerations especially in the first phase after the activation of the docking control system, an error weighing filter is used in the design. The authors showed with simulation that the proposed scheme satisfies the requirements for autonomous UAV in-flight refueling.

Campa et al [14] used the same fuzzy fusion strategy as that suggested by Fravolini et al [19] in their design of autonomous aerial refueling system with Boom and Receptacle method. In order to determine the relative position of the tanker aircraft with respect to UAV with machine vision system, markers were put on the tanker. In the study, the UAV dynamics is described by a linearized model. A reference trajectory is created when the AAR "tracking & docking" is activated at time t_0. The trajectory is expressed as a 3^{rd}order polynomial with respect to time t. The coefficients of the polynomial are evaluated by imposing desired boundary conditions. To ensure zero steady state tracking error for the position errors, the original linearized state space model is augmented with the integrals of the position errors. The design of the UAV docking control laws was formulated using a Linear Quadratic Regulator (LQR) approach.

For the autonomous aerial refueling system to work properly, an explicit knowledge of the position for all the markers is required by the machine vision system in order to find out the accurate relative position of the tanker. However, a small discrepancy of the real marker position with respect to the nominal design position might be possible due to effects such as tanker frame deformation. The effect of such discrepancy on the tanker position estimation error was simulated. The performance substantially deteriorates when the error on the exact location of all the markers is higher than 1%. The effects of the loss of visibility of one or more markers by the machine vision system were also studied. With a large enough initial set of optical markers properly located on the tanker aircraft, the estimation error does not seem to be substantially affected by the temporary loss of visibility.

Other control laws have also been proposed for the automatic flight control system for aerial refueling. Stepanyan, Lavrestky, and Hovakimyan [28] designed a control system for a receiver approaching and connecting up to the drogue with game theory, under the assumption that the position of the drogue can be measured by some method. Ochi and Kominami [25] observed that there are similarities between aerial refueling and missile guidance, where the proportional navigation guidance (PNG) is commonly used, and also approach guidance for instrument landing system, where line-of-sight (LOS) angle is precisely controlled. The observation led to flight control system design for automatic aerial refueling based on the PNG and the LOS angle control. In the PNG-based method, the longitudinal flight control system (FCS) controls upward acceleration and airspeed using the elevator and engine thrust, and the lateral-directional one controls side-ward acceleration and side-slip angle using aileron and rudder. In the LOS-angle-based method, the FCS controls integrals of flight path angle and flight directional angle along with the airspeed and side-slip angle. Simulation results show that both methods have good control performance under the circumstances without air turbulence. However, these methods may fail in the presence of turbulence.

It is believed that two of the most significant factors that affect the receiver aircraft's dynamics in aerial refueling are the time-varying inertia properties and the wind effect due to the tanker aircraft wake turbulence [15]. In the FCS designs mentioned above, the variation in receiver aircraft's mass is not considered. Pachter, Houpis, and Trosen [26] considered the variation of inertia properties in their design of an air-to-air automatic refueling flight control system. They considered mass variation using a quasi static method. The receiver aircraft is represented by sixteen models with different weights ranging from empty/low fuel to loaded/full fuel. A control system which is good for all sixteen models was designed. In this method, the dynamic effects of the inertia property variation were not considered.

Tanker aircraft wake turbulence is usually treated as disturbance in the controller design of FCS [32, 30, 26, 28, 25, 15, 14, 19, 21]. Dogan and Sato [15] designed a linear position-tracking controller with a combination of integral control and optimal LQR design similar to that of Campa et al. The controller does not use the information of the tanker aircraft's vortex induced wind effects acting on the receiver aircraft. To verify the performance of the controller, a set of nonlinear rigid body equations of motion for the receiver aircraft were derived. The nonlinear equations contain the wind effect terms and their time derivatives to represent the aerodynamic coupling between the two aircraft. These wind terms are obtained using an averaging technique [16].

7 Experimental Tests

Unmanned Air Vehicle has become an important asset in military operations. UAVs are invaluable in reconnaissance, target identification, target attack, and battle damage assessment. Autonomous aerial refueling extends the effectiveness of UAVs in several important ways. Challenges in UAV autonomous aerial refueling include [23]:

- Determination of the accurate relative position with tankers. The refueling procedure will require the UAV to operate in close proximity of the tanker aircraft. Therefore, it is critical for the UAV to know its accurate position with respect to the tanker aircraft.
- Collision avoidance. It is critical for the UAV to avoid collision with the tanker aircraft during aerial refueling procedure.
- Command and control. It is important for the UAV to respond to the boom operator's breakaway commands in the event an unsafe refueling condition occurs.
- Aircraft integration. Due to considerations relevant to cost, maintenance, availability, and constraints on weight and size, it is important to minimize the modifications to the tanker fleet and UAVs.
- Real-world constraints. AAR must be functional in all weather and day/night.

Flight tests and Man-in-the-loop simulation stations have been used to study the potential problems in UAV autonomous aerial refueling. One such man-in-the-loop system was developed by Burns et al [13].

As a part of the man-in-the-loop system, a prototype UAV control station interface for automated aerial refueling was developed by Williams et al [35]. It is used to control multiple unmanned air vehicles during the air refueling phase of flight. On the interface, the status of the tanker and UAVs are displayed. The operator creates high-level commands to move UAVs among different positions (observation, pre-contact, contact, post refueling) by mouse clicking. The high-level commands are sent to the UAVs. Having received high-level commands from the UAV operator, UAVs will generate corresponding lower-level trajectory commands for the UAV guidance, navigation and control systems to achieve the operator command objective. The prototype UAV control station interface is evaluated in a simulation environment with a KC-135 tanker and up to four UAVs simulated by computers. The interface performed satisfactorily though several issues are still to be resolved. In addition to the simulations on the man-in-the-loop system, flight testing has been used to verify the concept and possibility of automated aerial refueling of UAVs [24].

8 Conclusion

Demand for UAV autonomous aerial refueling capability has stimulated research activities in the area of aerial refueling. Aerial refueling research can be divided into four general areas: 1) Influence of tanker aircraft wake turbulence on the receiver aircraft; 2) the dynamics of the drogue and hose; 3) automatic flight control system design for aerial refueling; 4) experiments and flight tests related to the practical implementation of autonomous aerial refueling system.

Research work indicates that the tanker aircraft wake turbulence affects the stability and control of the receiver aircraft. The resulting forces and moments on the receiver aircraft can be predicted by either simplified models or by complicated CFD models. Researchers have studied the dynamics of the hose and drogue with FEA methods and experimental methods. Although progress has been made, it is still a challenge to accurately predict the position of the hose and drogue under the influence of the vortex field of the tanker wake turbulence and receiver aircraft nose. Such a prediction is actually impossible if random wind gust is assumed. Several automatic flight control systems with different control laws and position sensing methods have been proposed by researchers. Although most of the proposed flight control systems have been verified by simulations to satisfy design specifications, none of them has been verified by actual aerial refueling yet. UAV autonomous aerial refueling is still at such an early stage of implementation that there is no UAV available that is mature enough for modeling or actual flight experiments of aerial refueling. Other aircrafts (or models) are still used as “surrogate” UAV in simulations and flight tests.

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A New Approach for Construction of the Matrix-Valued Liapunov Functionals

A.A. Martynyuk * and V.I. Slyn'ko

Stability of Processes Departament,
S.P. Timoshenko Institute of Mechanics,
Nesterov str.3, 03057, Kiev-57, Ukraine

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Abstract: In this paper we analyze the stability of large-scale functional differential equations with constant delays via matrix-valued functionals of Liapunov-Krasovskii. We establish a new approach for construction of the Liapunov-Krasovskii functional and present conditions which guarantee the uniform asymptotic stability of the trivial solution of linear and quasi-linear functional differential equations.

Keywords: *Functional differential equations; Liapunov's functional; uniform asymptotic stability; oscillator with delay.*

Mathematics Subject Classification (2000): 34K20, 34K06, 34D20, 93D30.

1 Introduction

As is known, the direct Liapunov method [12] proves to be one of universal techniques of qualitative analysis of dynamical systems. Though the results achieved for the last decades in the development of this method (see [1, 4, 9, 15]) a series of general problems of motion stability theory shill remain in the focus of attention of many mathematicians and mechanical scientists. One of such problems is the problem of constructing suitable Liapunov functions (functionals) for certain classes of systems of equations.

For linear equations with constant coefficients and constant delay the problem on functional construction in [7, 6] is associated with solution of transcendent equations. Note that practical solution of the transcendent equation (see [7], p. 441)

$$\det(\lambda I - A - Be^{-\lambda\tau}) = 0, \tag{1.1}$$

* Corresponding author: anmart@stability.kiev.ua

may prove to be a problem difficult to be solved, especially in the case when the system under investigation is of large dimensions.

In the case when equation (1.1) is solved, stability of zero solution follows automatically and construction of the functional has the meaning of converse theorem.

In the context of our approach the methods of construction of functionals from [6, 7] etc. may be useful in the construction of diagonal elements of matrix-valued functional provided all necessary constants and comparison functions in their above and below estimates will satisfy appropriate conditions.

General conclusion of the carried out analysis is as follows. The approaches involving solution of transcendent equation are hardly applicable for the construction of functionals for large-scale delay systems. In the case of systems of small dimensions the proposed methods of functional constructions can be applied to construct diagonal elements of matrix-valued functional, however the methods in themselves can not solve the problem on stability of the initial large scale system.

The aim of this paper is to present a new method of constructing the Liapunov functionals for the class of linear delay systems. The method is based on the idea of approximation of functional differential equation by the system of difference equations (see [4]) in combination with the idea of application of matrix-valued Liapunov functional (see [16]). This allows one to extend the class of admissible functionals suitable for construction of the Liapunov functionals for the system of equations under consideration.

An auxiliary result in this paper is a method of constructing the matrix-valued function for a system of difference equations of larger dimensions (see [18]).

2 Notation and Assumptions

In this section we introduce main designations used in the paper and assumptions on the systems under consideration.

Let $r > 0$ be given and $\mathcal{C} = C([-r,0], R^n)$ be the space of continuous functions mapping $[-r,0]$ into R^n. For $\varphi \in \mathcal{C}$ we define the norm

$$\|\varphi\| = \sup_{-r\le\theta\le 0} |\varphi(\theta)|, \tag{2.1}$$

where $|\cdot|$ means the Euclidean norm in R^n. Let C_H be an open subset of C for the elements of which $\|\varphi\| < H$ and $0 \in C_H$. If $x:\ [-r,a) \to R^n$ and is continuous, $0 < a \le +\infty$, then for each $t \in [0,a)$ in C_H $x_t(s) = x(t+s)$, $-r \le s \le 0$. In addition to norm (2.1) we apply the norm

$$\|\varphi\|_{L_2} = \left\{ \int_{-r}^{0} |\varphi(\theta)|^2 d\,\theta \right\}^{1/2} \tag{2.2}$$

in the space $L_2([-r,0], R^n)$ of Lebesque functions integrated with square.

We study below the system with finite delay

$$\frac{dx}{dt} = F(x, x_t), \quad x_{t_0} = \varphi_0 \in \mathcal{C}, \quad t_0 \ge 0, \tag{2.3}$$

where $x \in R^n$, $F \in C(R^n \times \mathcal{C}, R^n)$, which has a linear approximation

$$\frac{dx}{dt} = Ax(t) + Bx(t-r) + f(x, x_t). \tag{2.4}$$

Here A and B are constant $n \times n$ matrices, $x(t)$ is n-dimensional vector, $r \geq 0$, $f \in C(R^n \times \mathcal{C}, R^n)$. The linear approximation of system (2.4)

$$\frac{dx}{dt} = Ax(t) + Bx(t-r) \tag{2.5}$$

is decomposed into two subsystems

$$\begin{aligned} \frac{dx_1}{dt} &= A_{11}x_1(t) + A_{12}x_2(t) + B_{11}x_1(t-r) + B_{12}x_2(t-r), \\ \frac{dx_2}{dt} &= A_{21}x_1(t) + A_{22}x_2(t) + B_{21}x_1(t-r) + B_{22}x_2(t-r), \end{aligned} \tag{2.6}$$

where $x_i \in R^{n_i}$, $i = 1, 2$, $(x_1^{\mathrm{T}}, x_2^{\mathrm{T}})^{\mathrm{T}} = x$, A_{ij} and B_{ij} are constant matrices of the appropriate dimensions for which the independent sybsystems are

$$\begin{aligned} \frac{dx_1}{dt} &= A_{11}x_1(t) + B_{11}x_1(t-r), \\ \frac{dx_2}{dt} &= A_{22}x_2(t) + B_{22}x_2(t-r). \end{aligned} \tag{2.7}$$

For system (2.6) the matrix-valued functional

$$U(x, \varphi(\cdot)) : \quad R^n \times \mathcal{C}^n \to R^{2\times 2} \tag{2.8}$$

is constructed of some scalar elements $v_{ij}(\varphi_1, \varphi_2)$, $i, j = 1, 2$, under additional assumptions on matrices A_{ii} and $B_{ii}, i = 1, 2$ of system (2.7).

The scalar functional (cf. [3])

$$v(x, \varphi, \eta) = \eta^{\mathrm{T}} U(x, \varphi(\cdot))\eta, \quad \eta \in R_+^2, \quad \eta > 0, \tag{2.9}$$

together with upper right derivative number [9]

$$D^+ v(x, \varphi, \eta)\big|_{(2.4)} : \quad R^n \times \mathcal{C} \to R$$

is the Liapunov–Krasovskii functional, if it solves the problem on stability of the state $x = 0$ of the system (2.4).

We recall that $D^+ v(x, \varphi, \eta)\big|_{(2.4)}$ is calculated by the formula

$$D^+ v(x, \varphi, \eta)\big|_{(2.4)} = \eta^{\mathrm{T}} D^+ U(x, \varphi)\big|_{(2.4)}\eta,$$

where

$$D^+ U(x, \varphi)\big|_{(2.4)} = \limsup_{\theta \to 0^+} \{[U(x + \theta F(x, \varphi), \varphi(0) + \theta F(x, \varphi)) - U(x, \varphi(0))]\theta^{-1}\}$$

is calculated element-wise. Before we start the solution of the problem on construction of functional (2.8) we need some results for the system of difference equations.

3 An Approach to Construction of Liapunov-Krasovskii Functionals

Now we consider autonomous linear system (2.5). Assume that for subsystems (2.7) the functionals

$$\begin{aligned} v_{11}(\varphi_1) &= \varphi_1^{\mathrm T}(0)P_{11}\varphi_1(0) + 2\varphi_1^{\mathrm T}(0)\int_{-r}^{0} K_1(\theta)\varphi_1(\theta)\,d\theta \\ &+ \int_{-r}^{0} \varphi_1^{\mathrm T}(\theta)\Gamma_1(\theta)\varphi_1(\theta)\,d\theta + \int_{-r}^{0}\int_{-r}^{0} \varphi_1^{\mathrm T}(\xi)\gamma_1(\xi,\eta)\varphi_1(\eta)\,d\xi\,d\eta, \\ v_{22}(\varphi_2) &= \varphi_2^{\mathrm T}(0)P_{22}\varphi_2(0) + 2\varphi_2^{\mathrm T}(0)\int_{-r}^{0} K_2(\theta)\varphi_2(\theta)\,d\theta \\ &+ \int_{-r}^{0} \varphi_2^{\mathrm T}(\theta)\Gamma_2(\theta)\varphi_2(\theta)\,d\theta + \int_{-r}^{0}\int_{-r}^{0} \varphi_2^{\mathrm T}(\xi)\gamma_2(\xi,\eta)\varphi_2(\eta)\,d\xi\,d\eta, \end{aligned} \tag{3.1}$$

are constructed somehow, where P_{11}, P_{22} are constant symmetric positive definite matrices,

$$\begin{aligned} &K_1,\Gamma_1 \in C([-r,0],\, R^{n_1\times n_1}), && K_2,\Gamma_2 \in C([-r,0],\, R^{n_2\times n_2}),\\ &\gamma_1 \in C([-r,0]\times[-r,0],\, R^{n_1\times n_1}), && \gamma_2 \in C([-r,0]\times[-r,0],\, R^{n_2\times n_2}). \end{aligned}$$

Further we employ the idea of approximation of system (2.6) by system of difference equations. With this in mind we divide the segment $[-r,0]$ into N equal parts of length h, i.e. $Nh = r$; the derivatives $\dfrac{dx_i}{dt}$, $i = 1,2$, are approximated by the differences $(x_i(t+h)-x_i(t))h^{-1}$. The system of difference equations corresponding to system (2.7) is (cf. Hale [4])*

$$\begin{aligned} &\widetilde x_{11}(\tau+1) = (I_{n_1}+hA_{11})\widetilde x_{11}(\tau) + hB_{11}\widetilde x_{1N}(t) + hA_{12}\widetilde x_{21}(t) + hB_{12}\widetilde x_{2N}(t),\\ &\widetilde x_{12}(\tau+1) = \widetilde x_{11}(\tau),\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &\widetilde x_{1N}(\tau+1) = \widetilde x_{1N-1}(\tau),\\ &\widetilde x_{21}(\tau+1) = (I_{n_2}+hA_{22})\widetilde x_{21} + hB_{22}\widetilde x_{2N}(\tau) + hA_{12}\widetilde x_{21}(\tau) + hB_{12}\widetilde x_{2N}(\tau),\\ &\widetilde x_{22}(\tau+1) = \widetilde x_{21}(\tau),\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &\widetilde x_{2N}(\tau+1) = \widetilde x_{2N-1}(\tau), \end{aligned} \tag{3.2}$$

where I_{n_1}, I_{n_2} are identity matrices of the corresponding dimensions.

The point $\widetilde x_i(0) = (\varphi_i(0),\, \varphi_i(-h),\, \dots,\, \varphi_i(-Nh))^{\mathrm T}$ corresponds to the initial function specifying solution $\widetilde x = (\widetilde x_1^{\mathrm T}, \widetilde x_2^{\mathrm T})^{\mathrm T}$ of system of difference equations (3.2).

Further we present system (3.2) in matrix form

$$\begin{aligned} \widetilde x_1(\tau+1) &= A_{11}\widetilde x_1(\tau) + A_{12}\widetilde x_2(\tau),\\ \widetilde x_2(\tau+1) &= A_{21}\widetilde x_1(\tau) + A_{22}\widetilde x_2(\tau), \end{aligned} \tag{3.3}$$

* It should be noted here that for stability analysis of the zero solution of system (2.3) with decomposition (2.6) a formal approach presented by Hale [4], p. 138 – 141, is employed.

where $\widetilde{x}_1 \in R^{n_1(N+1)}$, $\widetilde{x}_2 \in R^{n_2(N+1)}$ and

$$\widetilde{A}_{11} = \begin{pmatrix} I_{n_1} + hA_{11} & O_{n_1} & \dots & O_{n_1} & hB_{11} \\ I_{n_1} & O_{n_1} & \dots & O_{n_1} & O_{n_1} \\ \dots & \dots & \dots & \dots & \dots \\ O_{n_1} & \dots & \dots & I_{n_1} & O_{n_1} \end{pmatrix},$$

$$\widetilde{A}_{22} = \begin{pmatrix} I_{n_2} + hA_{22} & O_{n_2} & \dots & O_{n_2} & hB_{22} \\ I_{n_2} & O_{n_2} & \dots & O_{n_2} & O_{n_2} \\ \dots & \dots & \dots & \dots & \dots \\ O_{n_2} & \dots & \dots & I_{n_2} & O_{n_2} \end{pmatrix},$$

$$\widetilde{A}_{12} = \begin{pmatrix} hA_{12} & O_{n_1 \times n_2} & \dots & hB_{12} \\ O_{n_1 \times n_2} & O_{n_1 \times n_2} & \dots & O_{n_1 \times n_2} \\ \dots & \dots & \dots & \dots \\ O_{n_1 \times n_2} & \dots & \dots & O_{n_1 \times n_2} \end{pmatrix},$$

$$\widetilde{A}_{21} = \begin{pmatrix} hA_{21} & O_{n_2 \times n_1} & \dots & hB_{21} \\ O_{n_2 \times n_1} & O_{n_2 \times n_1} & \dots & O_{n_2 \times n_1} \\ \dots & \dots & \dots & \dots \\ O_{n_2 \times n_1} & \dots & \dots & O_{n_2 \times n_1} \end{pmatrix}.$$

Let k be arbitrary number, then vector $(\widetilde{x}_1^{\mathrm{T}}(kh),\, x_2^{\mathrm{T}}(kh))^{\mathrm{T}}$ is a phase vector for system (3.3) for any $t = kh$. For sufficiently small h vector $\widetilde{x}_i(kh)$, $i = 1, 2,$ is an exact enough approximation of solutions of system (2.7) at points kh, $k = 0, -1, \dots, -N$.

Functionals $v_{11}(\varphi_1)$ and $v_{22}(\varphi_2)$ are approximated by the quadratic forms

$$\widetilde{v}_{11}(\widetilde{x}_1) = \widetilde{x}_1^{\mathrm{T}} \widetilde{P}_{11} \widetilde{x}_1, \quad \widetilde{v}_{22}(\widetilde{x}_2) = \widetilde{x}_2^{\mathrm{T}} \widetilde{P}_{22} \widetilde{x}_2, \tag{3.4}$$

where

$$\widetilde{P}_{11} = \begin{pmatrix} NP_{11} & k_{11}^{\mathrm{T}} & k_{12}^{\mathrm{T}} & \dots & k_{1N}^{\mathrm{T}} \\ k_{11} & \alpha_{11}^1 & \alpha_{12}^1 & \dots & \alpha_{1N}^1 \\ k_{12} & \alpha_{12}^1 & \alpha_{22}^1 & \dots & \alpha_{2N}^1 \\ \dots & \dots & \dots & \dots & \dots \\ k_{1N} & \alpha_{1N}^1 & \alpha_{2N}^1 & \dots & \alpha_{NN}^1 \end{pmatrix},$$

$$\widetilde{P}_{22} = \begin{pmatrix} NP_{22} & k_{21}^{\mathrm{T}} & k_{22}^{\mathrm{T}} & \dots & k_{2N}^{\mathrm{T}} \\ k_{21} & \alpha_{11}^2 & \alpha_{12}^2 & \dots & \alpha_{1N}^2 \\ k_{22} & \alpha_{12}^2 & \alpha_{22}^2 & \dots & \alpha_{2N}^2 \\ \dots & \dots & \dots & \dots & \dots \\ k_{2N}^2 & \alpha_{1N}^2 & \alpha_{2N}^2 & \dots & \alpha_{NN}^2 \end{pmatrix}.$$

Here the constant matrices $P_{11}, P_{22}, k_{ji}, \alpha_{ij}^j$, $i = 1, 2, \dots, N$, $j = 1, 2,$ of the corresponding dimensions are determined as

$$k_{ji} = K_j(-hi), \quad \alpha_{ii}^j = \Gamma_j(-ih), \quad i = 1, 2, \dots, N, \quad j = 1, 2,$$
$$\alpha_{ij}^1 = \gamma_1(-hi, -hj), \quad \alpha_{ij}^2 = \gamma_2(-hi, -hj), \quad i, j = 1, 2, \dots, N, \quad i \neq j.$$

We construct the non-diagonal element $v_{12}(\widetilde{x}_1, \widetilde{x}_2)$ of the matrix-valued functional $U(\widetilde{x}_1, \widetilde{x}_2)$ in the bilinear form

$$v_{12}(\widetilde{x}_1, \widetilde{x}_2) = \widetilde{x}_1^{\mathrm{T}} \widetilde{P}_{12} \widetilde{x}_2, \tag{3.5}$$

where matrix $\widetilde{P}_{12}$ satisfies the equation

$$\widetilde{A}_{11}^{\mathrm{T}}\widetilde{P}_{12}\widetilde{A}_{22} - \widetilde{P}_{12} = -\frac{\eta_1}{\eta_2}\widetilde{A}_{11}^{\mathrm{T}}\widetilde{P}_{11}\widetilde{A}_{12} - \frac{\eta_2}{\eta_1}\widetilde{A}_{21}^{\mathrm{T}}\widetilde{P}_{22}\widetilde{A}_{22} \tag{3.6}$$

and has the form

$$\widetilde{P}_{12} = \begin{pmatrix} NP_{12} & s_1^2 & s_2^2 & \dots & s_N^2 \\ s_1^1 & q_{11} & q_{12} & \dots & q_{1N} \\ s_2^1 & q_{21} & q_{22} & \dots & q_{2N} \\ \dots & \dots & \dots & \dots & \dots \\ s_N^1 & q_{N1} & q_{N2} & \dots & q_{NN} \end{pmatrix}.$$

Here P_{12}, s_i^j, q_{ij} are matrices and η_1, η_2 are positive constants.

In terms of equation (3.6) we get

$$\begin{aligned}(NP_{12} + rA_{11}^{\mathrm{T}}P_{12} + s_1^1)(I_{n_2} + hA_{22}) + s_1^2 + hA_{11}^{\mathrm{T}}s_1^2 + q_{11} - NP_{12} \\ = -\frac{r\eta_1}{\eta_2}P_{11}A_{12} - hr\frac{\eta_1}{\eta_2}A_{11}P_{11}A_{12} - h\frac{\eta_1}{\eta_2}k_{11}A_{11}k_{1N}^{\mathrm{T}}A_{12} \\ - r\frac{\eta_2}{\eta_1}A_{21}^{\mathrm{T}}P_{22} - r\frac{\eta_1}{\eta_2}hA_{21}^{\mathrm{T}}P_{22}A_{22} - \frac{\eta_2}{\eta_1}hA_{21}^{\mathrm{T}}k_{2N}A_{21}^{\mathrm{T}}k_{21}^{\mathrm{T}},\end{aligned} \tag{3.7}$$

$$s_i^2 - s_{i-1}^2 + hA_{11}^{\mathrm{T}}s_i^2 + q_{1i} = -\frac{\eta_2}{\eta_1}hA_{21}^{\mathrm{T}}k_{2i}, \quad i = 2, \dots, N, \tag{3.8}$$

$$hs_i^1B_{22} - q_{i-1,N} = -\frac{\eta_1}{\eta_2}hk_{1i}^{\mathrm{T}}B_{12}, \quad i = 2, \dots, N, \tag{3.9}$$

$$hB_{11}^{\mathrm{T}}s_i^2 - q_{N,i-1} = -\frac{\eta_2}{\eta_1}hB_{21}^{\mathrm{T}}k_{2i}, \quad i = 2, \dots, N, \tag{3.10}$$

$$\begin{aligned}rP_{12}B_{22} + hrA_{11}^{\mathrm{T}}P_{12}B_{22} + hs_1^1B_{22} - s_N^2 = -\frac{\eta_1}{\eta_2}rP_{11}B_{12} \\ - \frac{r\eta_1}{\eta_2}hA_{11}^{\mathrm{T}}P_{11}B_{12} - \frac{\eta_1}{\eta_2}hk_{11}B_{12} - \frac{r\eta_2}{\eta_1}hA_{21}^{\mathrm{T}}P_{22}B_{22},\end{aligned} \tag{3.11}$$

$$\begin{aligned}rB_{11}^{\mathrm{T}}P_{12} + hrB_{11}^{\mathrm{T}}P_{12}A_{22} + hB_{11}^{\mathrm{T}}s_1^2 - s_N^1 = -\frac{r\eta_1}{\eta_2}hB_{11}^{\mathrm{T}}P_{11}A_{12} \\ - \frac{r\eta_2}{\eta_1}B_{21}^{\mathrm{T}}P_{22} - \frac{r\eta_2}{\eta_1}hB_{21}^{\mathrm{T}}P_{22}A_{22} - \frac{\eta_2}{\eta_1}hB_{21}^{\mathrm{T}}k_{12}^{\mathrm{T}},\end{aligned} \tag{3.12}$$

$$s_i^1 + hs_i^1A_{22} - s_{i-1}^1 + q_{i1} = -hk_{1i}A_{12}, \quad i = 2, \dots, N, \tag{3.13}$$

$$q_{NN} = h\left(B_{11}^{\mathrm{T}}B_{22} + \frac{\eta_1}{\eta_2}B_{11}B_{12} + \frac{\eta_2}{\eta_1}B_{21}B_{22}\right), \tag{3.14}$$

$$q_{ii} = \mathrm{const}, \quad q_{ij} = q_{i-1,j-1}, \quad i, j = 2, \dots, N. \tag{3.15}$$

From equation (3.7), in view of (3.4) and $h \to 0$ we get*

$$A_{11}^{\mathrm{T}}P_{12} + P_{12}A_{22} = -\frac{\eta_1}{\eta_2}P_{11}A_{12} - \frac{\eta_2}{\eta_1}A_{21}^{\mathrm{T}}P_{22} - \frac{1}{r}\left(S_1(0) + S_2(0)\right). \tag{3.16}$$

Similarly in view of (3.15) we get from (3.9)

$$q_{1i} = q_{N+1-i,N} = h(s_{N+2-i}^1B_{22} + k_{1,N+2-i}^{\mathrm{T}}B_{12}). \tag{3.17}$$

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* From here on in formulas (3.7) – (3.14) and (3.18) passage to the limit as $h \to 0$ is formal.

Then, in view of (3.17) equations (3.9) imply

$$(s_i^2 - s_{i-1}^2)h^{-1} + A_{11}^{\mathrm{T}} s_i^2 + s_{N+2-i}^1 B_{22} + \frac{\eta_1}{\eta_2} k_{1,N+2-i} B_{12} = -\frac{\eta_2}{\eta_1} A_{21}^{\mathrm{T}} k_{2i}^{\mathrm{T}}, \quad i = 2, \dots, N, \tag{3.18}$$

and passing to the limit as $h \to 0$ we obtain

$$-\frac{dS_2}{d\theta} + A_{11}^{\mathrm{T}} S_2(\theta) + S_1(-r-\theta) B_{22} + \frac{\eta_1}{\eta_2} K_1(-r-\theta) B_{12} = -\frac{\eta_2}{\eta_1} A_{21}^{\mathrm{T}} K_2^{\mathrm{T}}(\theta). \tag{3.19}$$

Similarly to the above, in view of (3.14) we get from (3.19)

$$-\frac{dS_1}{d\theta} + S_1(\theta) A_{22} + B_{11}^{\mathrm{T}} S_2(-r-\theta) + \frac{\eta_2}{\eta_1} B_{21}^{\mathrm{T}} K_2^{\mathrm{T}}(-r-\theta) = -\frac{\eta_1}{\eta_2} K_1(\theta) A_{12}. \tag{3.20}$$

Taking into account (3.14) we find from (3.11) and (3.12) as $h \to 0$ the initial conditions

$$\begin{aligned} S_2(-1) &= r\left(P_{12} B_{22} + \frac{\eta_1}{\eta_2} P_{11} B_{12}\right), \\ S_1(-1) &= r\left(B_{11}^{\mathrm{T}} P_{12} + \frac{\eta_2}{\eta_1} B_{21}^{\mathrm{T}} P_{22}\right). \end{aligned} \tag{3.21}$$

In the expression of the bilinear form $\frac{1}{N} v_{12}(\widetilde{x}_1, \widetilde{x}_2)$ the formal limiting passage $(h \to 0)$ yields the expression for the functional

$$\begin{aligned} v_{12}(\varphi_1, \varphi_2) &= \varphi_1^{\mathrm{T}}(0) P_{12} \varphi_2(0) + \frac{1}{r} \varphi_1^{\mathrm{T}}(0) \int_{-r}^{0} S_2(\theta) \varphi_2(\theta)\, d\theta \\ &+ \frac{1}{r} \varphi_2^{\mathrm{T}}(0) \int_{-r}^{0} S_1^{\mathrm{T}}(\theta) \varphi_1(\theta)\, d\theta + \frac{1}{r} \int_{-r}^{0} d\xi \varphi_1^{\mathrm{T}}(\xi) \int_{-r}^{\xi} \Big\{ S_1(\xi - \eta - r) B_{22} \\ &+ \frac{\eta_1}{\eta_2} K_1^{\mathrm{T}}(\xi - \eta - r) B_{12} \Big\} \varphi_2(\eta)\, d\eta + \frac{1}{r} \int_{-r}^{0} d\xi \varphi_1^{\mathrm{T}}(\xi) \int_{\xi}^{0} \Big\{ B_{11}^{\mathrm{T}} S_2(\eta - \xi - r) \\ &+ \frac{\eta_1}{\eta_2} B_{21}^{\mathrm{T}} K_2(\eta - \xi - r) \Big\} \varphi_2(\eta) d\eta. \end{aligned} \tag{3.22}$$

In order to formulate stability conditions for system (2.6) in terms of the matrix-valued functional $U(\varphi_1, \varphi_2)$ with components (3.1) and (3.22) it is necessary to estimate its and their upper right derivative numbers along solutions of system (2.6). To this end we define concretely the choice of functionals (3.1) as

$$v_{11}(\varphi_1) = \varphi_1^{\mathrm{T}}(0) P_{11} \varphi_1(0) + \int_{-r}^{0} k(\theta) \varphi_1^{\mathrm{T}}(\theta) D_1 \varphi_1(\theta)\, d\theta, \tag{23}$$

$$v_{22}(\varphi_2) = \varphi_2^{\mathrm{T}}(0) P_{22} \varphi_2(0) + \int_{-r}^{0} k(\theta) \varphi_2^{\mathrm{T}}(\theta) D_2 \varphi_2(\theta)\, d\theta, \tag{24}$$

where P_{11}, P_{22}, D_1 and D_2 are positive definite matrices of the corresponding dimensions and $k(\theta) = 1 + \dfrac{1}{2r}\theta$.

Basing on the system of equations

$$\frac{dS_2}{d\theta} = A_{11}^{\mathrm T} S_2(\theta) + S_1(-r-\theta)B_{22}, \tag{25}$$

$$\frac{dS_1}{d\theta} = S_1(\theta)A_{22} + B_{11}^{\mathrm T} S_2(-r-\theta), \tag{26}$$

$$A_{11}^{\mathrm T}P_{12} + P_{12}A_{22} = -\frac{\eta_1}{\eta_2}P_{11}A_{12} - \frac{\eta_2}{\eta_1}A_{21}^{\mathrm T}P_{22} - \frac{1}{r}(S_1(0) + S_2(0)) \tag{27}$$

under initial conditions

$$\begin{aligned} S_2(-r) &= r\Big(P_{12}B_{22} + \frac{\eta_1}{\eta_2}P_{11}B_{12}\Big),\\ S_1(-r) &= r\Big(B_{11}^{\mathrm T}P_{12} + \frac{\eta_2}{\eta_1}B_{21}^{\mathrm T}P_{22}\Big), \end{aligned} \tag{3.28}$$

where $P_{12} \in R^{n_1\times n_2}$, $S_1, S_2 \in C^1([-r,0], R^{n_1\times n_2})$, η_1, η_2 are positive constants we construct functional $v_{12}(\varphi_1,\varphi_2)$ in the form

$$\begin{aligned} v_{12}(\varphi_1,\varphi_2) &= \varphi_1^{\mathrm T}(0)P_{12}\varphi_2(0) + \frac{1}{r}\varphi_1^{\mathrm T}(0)\int_{-r}^{0} S_2(\theta)\varphi_2(\theta)\,d\theta\\ &+ \frac{1}{r}\varphi_2^{\mathrm T}(0)\int_{-r}^{0} S_1^{\mathrm T}(\theta)\varphi_1(\theta)\,d\theta + \frac{1}{r}\int_{-r}^{0} d\xi\,\varphi_1^{\mathrm T}(\xi)\int_{-r}^{\xi} S_1(\xi-\eta-r)B_{22}\varphi_2(\eta)\,d\eta\\ &+ \frac{1}{r}\int_{-r}^{0} d\xi\,\varphi_1^{\mathrm T}(\xi)\int_{\xi}^{0} B_{11}^{\mathrm T}S_2(\eta-\xi-r)\varphi_2(\eta)\,d\eta. \end{aligned} \tag{3.29}$$

Since for the functionals $v_{ij}(\cdot)$, $i,j = 1,2$, the lower estimates

$$\begin{aligned} v_{11}(\varphi_1) &\ge \lambda_m(P_{11})|\varphi_1(0)|^2 + \frac{1}{2}\lambda_m(D_1)\|\varphi_1\|_{L_2}^2\\ v_{22}(\varphi_2) &\ge \lambda_m(P_{22})|\varphi_2(0)|^2 + \frac{1}{2}\lambda_m(D_2)\|\varphi_2\|_{L_2}^2\\ v_{12}(\varphi_1,\varphi_2) &\ge -\|P_{12}\||\varphi_1(0)||\varphi_2(0)| - \varkappa_2|\varphi_1(0)|\|\varphi_2\|_{L_2}\\ &\quad - \varkappa_1|\varphi_2(0)|\|\varphi_1\|_{L_2} - (\varkappa_{21}\|B_{11}\| + \varkappa_{12}\|B_{22}\|)\|\varphi_1\|_{L_2}\|\varphi_2\|_{L_2}, \end{aligned} \tag{3.30}$$

and the upper estimates

$$\begin{aligned} v_{11}(\varphi_1) &\le \lambda_M(P_{11})|\varphi_1(0)|^2 + \frac{1}{2}\lambda_M(D_1)\|\varphi_1\|_{L_2}^2\\ v_{22}(\varphi_2) &\le \lambda_M(P_{22})|\varphi_2(0)|^2 + \frac{1}{2}\lambda_M(D_2)\|\varphi_1\|_{L_2}^2\\ v_{12}(\varphi_1,\varphi_2) &\le \|P_{12}\||\varphi_1(0)||\varphi_2(0)| + \varkappa_2|\varphi_1(0)|\|\varphi_2\|_{L_2}\\ &\quad + \varkappa_{12}|\varphi_2(0)|\|\varphi_1\|_{L_2} + (\varkappa_{21}\|B_{11}\| + \varkappa_{12}\|B_{22}\|)\|\varphi_1\|_{L_2}\|\varphi_2\|_{L_2}, \end{aligned} \tag{3.31}$$

are satisfied, where

$$\varkappa_1 = \frac{1}{r}\left\{\int_{-r}^{0}\|S_1(\theta)\|^2 d\theta\right\}^{1/2}, \quad \varkappa_2 = \frac{1}{r}\left\{\int_{-r}^{0}\|S_2(\theta)\|^2 d\theta\right\}^{1/2},$$

$$\varkappa_{12} = \frac{1}{r}\left\{\int_{-r}^{0}\int_{-r}^{0}\|S_1(\xi-\eta-r)\|^2 d\xi d\eta\right\}^{1/2},$$

$$\varkappa_{21} = \frac{1}{r}\left\{\int_{-r}^{0}\int_{-r}^{0}\|S_2(\xi-\eta-r)\|^2 d\xi d\eta\right\}^{1/2},$$

for the functional

$$v(\varphi_1,\varphi_2,\eta) = \eta^{\mathrm{T}}U(\varphi_1,\varphi_2)\eta = \eta_1^2 v_{11}(\varphi_1) + 2\eta_1\eta_2 v_{12}(\varphi_1,\varphi_2) + \eta_2^2 v_{22}(\varphi_2)$$

the bilateral estimate

$$u^{\mathrm{T}}H^{\mathrm{T}}\underline{C}Hu \le v(\varphi_1,\varphi_2,\eta) \le u^{\mathrm{T}}H^{\mathrm{T}}\overline{C}Hu, \tag{3.32}$$

is valid, where

$$u = (|\varphi_1(0)|,\ |\varphi_2(0)|,\ \|\varphi_1\|_{L_2},\ \|\varphi_2\|_{L_2}),$$
$$H = \mathrm{diag}(\eta_1,\eta_2,\eta_1,\eta_2), \quad \zeta = \varkappa_{21}\|B_{11}\| + \varkappa_{12}\|B_{22}\|,$$
$$\overline{C} = \begin{pmatrix} \lambda_M(P_{11}) & \|P_{12}\| & 0 & \varkappa_2 \\ \|P_{12}\| & \lambda_M(P_{22}) & \varkappa_1 & 0 \\ 0 & \varkappa_1 & \lambda_M(D_1) & \zeta \\ \varkappa_2 & 0 & \zeta & \lambda_M(D_2) \end{pmatrix},$$
$$\underline{C} = \begin{pmatrix} \lambda_m(P_{11}) & -\|P_{12}\| & 0 & -\varkappa_2 \\ -\|P_{12}\| & \lambda_m(P_{22}) & -\varkappa_1 & 0 \\ 0 & -\varkappa_1 & \frac{1}{2}\lambda_m(D_1) & -\zeta \\ -\varkappa_2 & 0 & -\zeta & \frac{1}{2}\lambda_m(D_2) \end{pmatrix}.$$

Further together with functionals (3.23), (3.24) and (3.29) we use the upper right derivative numbers $D^+v_{ij}(\cdot)\big|_{(2.6)}$, $i,j=1,2$:

$$D^+v_{11}(\varphi_1)\big|_{(2.6)} = \varphi_1^{\mathrm{T}}(0)(A_{11}^{\mathrm{T}}P_{11} + P_{11}A_{11} + D_1)\varphi_1(0)$$
$$-\frac{1}{2}\varphi_1^{\mathrm{T}}(-r)D_1\varphi_1(-r) + \varphi_1^{\mathrm{T}}(0)P_{11}B_{11}\varphi_1(-r) + \varphi_1^{\mathrm{T}}(0)P_{11}A_{12}\varphi_2(0) \tag{33}$$
$$+\varphi_1^{\mathrm{T}}(0)P_{11}B_{12}\varphi_2(-r) - \frac{1}{2r}\int_{-r}^{0}\varphi_1^{\mathrm{T}}(\theta)D_1\varphi_1(\theta)\,d\theta,$$

$$D^+v_{22}(\varphi_2)\big|_{(2.6)} = \varphi_2^{\mathrm{T}}(0)(A_{22}^{\mathrm{T}}P_{22} + P_{22}A_{22} + D_2)\varphi_2(0)$$
$$-\frac{1}{2}\varphi_2^{\mathrm{T}}(-r)D_1\varphi_2(-r) + \varphi_2^{\mathrm{T}}(0)P_{22}B_{22}\varphi_2(-r) + \varphi_2^{\mathrm{T}}(0)P_{22}A_{21}\varphi_1(0) \tag{34}$$

$$+ \varphi_2^{\mathrm T}(0)P_{22}B_{21}\varphi_1(-r) - \frac{1}{2r}\int\limits_{-r}^{0}\varphi_2^{\mathrm T}(\theta)D_2\varphi_2(\theta)\,d\theta,$$

$$\begin{aligned}
D^+v_{12}(\varphi_1,\varphi_2)\big|_{(2.6)} &= \varphi_1^{\mathrm T}(0)\Big(A_{11}^{\mathrm T}P_{12} + P_{12}A_{22} + \frac{1}{r}(S_1(0)+S_2(0))\Big)\varphi_2(0)\\
&+ \frac{1}{2}\varphi_1^{\mathrm T}(0)(P_{12}A_{21} + A_{21}^{\mathrm T}P_{12}^{\mathrm T})\varphi_1(0) + \frac{1}{2}\varphi_2^{\mathrm T}(0)(A_{12}^{\mathrm T}P_{12} + P_{12}^{\mathrm T}A_{12})\varphi_2(0)\\
&+ \varphi_1^{\mathrm T}(0)P_{12}B_{21}\varphi_1(-r) + \varphi_2^{\mathrm T}(-r)B_{12}^{\mathrm T}P_{12}\varphi_2(0)\\
&+ \frac{1}{r}\varphi_2^{\mathrm T}(-r)B_{12}^{\mathrm T}\int\limits_{-r}^{0}S_2(\theta)\varphi_2(\theta)\,d\theta + \frac{1}{r}\varphi_2^{\mathrm T}(0)A_{12}^{\mathrm T}\int\limits_{-r}^{0}S_2(\theta)\varphi_2(\theta)\,d\theta \qquad (35)\\
&+ \frac{1}{r}\varphi_1^{\mathrm T}(0)A_{21}^{\mathrm T}\int\limits_{-r}^{0}S_1^{\mathrm T}(\theta)\varphi_1(\theta)\,d\theta + \frac{1}{r}\varphi_1^{\mathrm T}(-r)B_{21}^{\mathrm T}\int\limits_{-r}^{0}S_1^{\mathrm T}(\theta)\varphi_1(\theta)\,d\theta\\
&- \frac{\eta_1}{\eta_2}\varphi_1^{\mathrm T}(0)P_{11}B_{12}\varphi_2(-r) - \frac{\eta_2}{\eta_1}\varphi_1^{\mathrm T}(-r)B_{21}^{\mathrm T}P_{22}\varphi_2(0).
\end{aligned}$$

In view of expressions (3.33) – (3.35) for the upper right derivative number of functional $v(\varphi_1,\varphi_2,\eta)$ in the domain of values $R^n\times\mathcal{C}^n$ we have the estimate

$$D^+v(\varphi_1,\varphi_2,\eta)\big|_{(2.6)} \le u_1^{\mathrm T}\Sigma_1u_1 + u_2^{\mathrm T}\Sigma_2u_2, \tag{3.36}$$

where

$$u_1 = (|\varphi_1(0)|,\ |\varphi_1(-r)|,\ \|\varphi_1\|_{L_2})^{\mathrm T},$$
$$u_2 = (|\varphi_2(0)|,\ |\varphi_2(-r)|,\ \|\varphi_2\|_{L_2})^{\mathrm T}$$

and $\Sigma_1 = [\sigma_{ij}^1]_{i,j=1}^3$, $\Sigma_2 = [\sigma_{ij}^2]_{i,j=1}^3$ are constant matrices with the elements

$$\begin{gathered}
\sigma_{11}^1 = \lambda_M(A_{11}^{\mathrm T}P_{11} + P_{11}A_{11} + D_1)\eta_1^2 + \eta_1\eta_2\lambda_M(P_{12}A_{21} + A_{21}^{\mathrm T}P_{12}^{\mathrm T}),\\
\sigma_{22}^1 = -\frac{1}{2}\lambda_m(D_1)\eta_1^2, \quad \sigma_{33}^1 = -\frac{1}{2r}\lambda_m(D_1)\eta_1^2,\\
\sigma_{12}^1 = \|P_{11}\|\,\|B_{11}\|\eta_1^2 + \|P_{12}\|\,\|B_{21}\|\eta_1\eta_2,\\
\sigma_{23}^1 = \varkappa_1\|B_{21}\|\eta_1\eta_2, \quad \sigma_{13}^1 = \varkappa_1\|A_{21}\|\eta_1\eta_2,\\
\sigma_{11}^2 = \lambda_M(A_{22}^{\mathrm T}P_{22} + P_{22}A_{22} + D_2)\eta_2^2 + \eta_1\eta_2\lambda_M(P_{21}A_{12} + A_{12}^{\mathrm T}P_{21}^{\mathrm T}),\\
\sigma_{22}^2 = -\frac{1}{2}\lambda_m(D_2)\eta_2^2, \quad \sigma_{33}^2 = -\frac{1}{2r}\lambda_m(D_2)\eta_2^2,\\
\sigma_{12}^2 = \|P_{22}\|\,\|B_{22}\|\eta_2^2 + \|P_{12}\|\,\|B_{12}\|\eta_1\eta_2,\\
\sigma_{23}^2 = \varkappa_2\|B_{12}\|\eta_1\eta_2, \quad \sigma_{13}^2 = \varkappa_2\|A_{12}\|\eta_1\eta_2,\\
\sigma_{ij}^1 = \sigma_{ji}^1, \quad \sigma_{ij}^2 = \sigma_{ji}^2, \quad i,j = 1,2,3, \quad i \ne j.
\end{gathered}$$

In the partial case when $B_{11} = 0$ and $B_{22} = 0$ system (2.6) becomes

$$\begin{aligned}
\frac{dx_1}{dt} &= A_{11}x_1(t) + A_{12}x_2(t) + B_{12}x_2(t-r),\\
\frac{dx_2}{dt} &= A_{21}x_1(t) + A_{22}x_2(t) + B_{21}x_1(t-r).
\end{aligned} \tag{3.37}$$

Besides, system of equations (3.25) – (3.27) becomes

$$\begin{aligned}\frac{dS_2}{d\theta} &= A_{11}^{\mathrm{T}} S_2(\theta), \qquad \frac{dS_1}{d\theta} = S_1(\theta) A_{22} \\ A_{11}^{\mathrm{T}} P_{12} + P_{12} A_{22} &= -\frac{\eta_1}{\eta_2} P_{11} A_{12} - \frac{\eta_2}{\eta_1} A_{21}^{\mathrm{T}} P_{22} - \frac{1}{r}(S_1(0) + S_2(0))\end{aligned} \tag{3.38}$$

under the initial conditions

$$S_2(-r) = \frac{r\eta_1}{\eta_2} P_{11} B_{12}, \quad S_1(-r) = \frac{r\eta_2}{\eta_1} B_{21}^{\mathrm{T}} P_{22}. \tag{3.39}$$

The first group of equations (3.38) can be integrated in the explicit form

$$\begin{aligned}S_1(\theta) &= \frac{r\eta_2}{\eta_1} B_{21}^{\mathrm{T}} P_{22} \exp\{A_{22}(\theta + r)\}, \\ S_2(\theta) &= \frac{r\eta_1}{\eta_2} \exp\{A_{11}^{\mathrm{T}}(\theta + r)\} P_{11} B_{12}.\end{aligned} \tag{3.40}$$

Letting $\theta = 0$ we find

$$\begin{aligned}S_1(0) &= \frac{r\eta_2}{\eta_1} B_{21}^{\mathrm{T}} P_{22} \exp\{A_{22} r\}, \\ S_2(0) &= \frac{r\eta_1}{\eta_2} \exp\{A_{11}^{\mathrm{T}} r\} P_{11} B_{12}.\end{aligned}$$

Therefore equation (3.27) becomes

$$\begin{aligned}A_{11}^{\mathrm{T}} P_{12} + P_{12} A_{22} = &-\frac{\eta_1}{\eta_2}(P_{11} A_{12} + \exp\{A_{11}^{\mathrm{T}} r\} P_{11} B_{12}) \\ &- \frac{\eta_2}{\eta_1}(A_{21}^{\mathrm{T}} P_{22} + B_{21}^{\mathrm{T}} P_{22} \exp\{A_{22} r\}).\end{aligned} \tag{3.41}$$

Necessary and sufficient existence conditions for unique solution of equation (3.41) follow from Lankaster [11].

Diagonal elements of the matrix-valued functional $U(\varphi_1, \varphi_2)$ are taken in the form of (3.23), (3.24) for $w_1(\varphi_1) = v_{11}(\varphi_1)$ and $w_2(\varphi_2) = v_{22}(\varphi_2)$, and non-diagonal element $w_{12}(\varphi_1, \varphi_2)$ is represented as

$$\begin{aligned}w_{12}(\varphi_1, \varphi_2) = \varphi_1^{\mathrm{T}}(0) P_{12} \varphi_2(0) &+ \frac{\eta_1}{\eta_2} \varphi_1^{\mathrm{T}}(0) \int_{-r}^{0} \exp\{A_{11}^{\mathrm{T}}(\theta + r)\} P_{11} B_{12} \varphi_2(\theta)\, d\theta \\ &+ \frac{\eta_2}{\eta_1} \varphi_2^{\mathrm{T}}(0) \int_{-r}^{0} B_{21}^{\mathrm{T}} P_{22} \exp\{A_{22}(\theta + r)\} \varphi_1(\theta)\, d\theta.\end{aligned} \tag{3.42}$$

For estimation of functional (3.42) we shall formulate one auxiliary result (see [2]).

Lemma 3.1 *Let A be a constant $n \times n$–matrix, then estimate*

$$\|\exp At\| \le e^{\Delta t} \sum_{k=0}^{n-1} \frac{1}{k!} (2t\|A\|)^k, \quad t \ge 0,$$

is valid, where $\Delta = \max\{\mathrm{Re}\,\lambda \;\, \lambda \in \sigma(A)\}$, $\sigma(A)$ is a spectrum of matrix A.

Using this result one can estimate $\|\exp At\|$ as follows. Let $\varepsilon > 0$ be a sufficiently small positive number. Consider function [17]

$$f(t) = e^{-\varepsilon t} \sum_{k=0}^{n-1} \frac{1}{k!} (2t\|A\|)^k, \quad t \geq 0.$$

In view of the fact that $f(t) \to 0$ as $t \to \infty$ we conclude that there exists $M_\varepsilon = \max\limits_{t \geq 0} f(t)$ and find the estimate

$$\|\exp\{At\}\| \leq M_\varepsilon e^{(\Delta+\varepsilon)t} \quad \text{for} \quad t \geq 0. \tag{3.43}$$

Applying estimate (3.43) it is easy to find

$$\varkappa_1 = \frac{1}{r}\left\{\int\limits_{-r}^{0} \lambda_M(S_1(\theta)S_1^{\mathrm{T}}(\theta))\right\}^{1/2} \leq \frac{\eta_2}{\eta_1}\|B_{21}\|\,\|P_{22}\|M_{\varepsilon 1}\left[\frac{e^{2(\Delta_1+\varepsilon)r}-1}{2(\Delta_1+\varepsilon)}\right]^{1/2} = \xi_1,$$

$$\varkappa_2 = \frac{1}{r}\left\{\int\limits_{-r}^{0} \lambda_M(S_2(\theta)S_2^{\mathrm{T}}(\theta))\right\}^{1/2} \leq \frac{\eta_1}{\eta_2}\|B_{12}\|\,\|P_{11}\|M_{\varepsilon 2}\left[\frac{e^{2(\Delta_2+\varepsilon)r}-1}{2(\Delta_2+\varepsilon)}\right]^{1/2} = \xi_2,$$

where $M_{\varepsilon 1}$ and $M_{\varepsilon 1}$ are the corresponding constants and Δ_1 and Δ_2 are maximal real values of spectra of matrices A_{11} and A_{22} respectively.

Thus, for the scalar functional

$$w(\varphi_1, \varphi_2, \eta) = \eta^{\mathrm{T}}\, U(\varphi_1, \varphi_2)\eta \tag{3.44}$$

the estimate

$$u^{\mathrm{T}} H^{\mathrm{T}} \underline{C} H u \leq w(\varphi_1, \varphi_2, \eta) \leq u^{\mathrm{T}} H^{\mathrm{T}}\, \overline{C} H u, \tag{3.45}$$

is valid, where

$$u = (|\varphi_1(0)|,\, |\varphi_2(0)|,\, \|\varphi_1\|_{L_2},\, \|\varphi_2\|_{L_2})^{\mathrm{T}}, \quad H = \mathrm{diag}(\eta_1, \eta_2, \eta_1, \eta_2),$$

$$\overline{C} = \begin{pmatrix} \lambda_M(P_{11}) & \|P_{12}\| & 0 & \xi_2 \\ \|P_{12}\| & \lambda_M(P_{22}) & \xi_1 & 0 \\ 0 & \xi_1 & \lambda_M(D_1) & 0 \\ \xi_2 & 0 & 0 & \lambda_M(D_2) \end{pmatrix},$$

$$\underline{C} = \begin{pmatrix} \lambda_m(P_{11}) & -\|P_{12}\| & 0 & -\xi_2 \\ -\|P_{12}\| & \lambda_m(P_{22}) & -\xi_1 & 0 \\ 0 & -\xi_1 & \frac{1}{2}\lambda_m(D_1) & 0 \\ -\xi_2 & 0 & 0 & \frac{1}{2}\lambda_m(D_2) \end{pmatrix}.$$

In view of estimates (3.36) and (3.43) in the region of values $R^n \times \mathcal{C}^n$ it is easy to find the estimate of the upper right derivative number of functional (3.44) along solutions of system (3.37)

$$D^+ v(\varphi_1, \varphi_2, \eta)\big|_{(3.37)} \leq u_1^{\mathrm{T}} \Omega_1 w_1 + u_2^{\mathrm{T}} \Omega_2 w_2, \tag{3.46}$$

and $\Omega_1 = [\omega^1_{ij}]^3_{i,j=1}$, $\Omega_2 = [\omega^2_{ij}]^3_{i,j=1}$ are constant matrices with the elements

$$\omega^1_{11} = \lambda_M(A^{\mathrm T}_{11}P_{11} + P_{11}A_{11} + D_1)\eta^2_1 + \eta_1\eta_2\lambda_M(P_{12}A_{21} + A^{\mathrm T}_{21}P^{\mathrm T}_{12}),$$
$$\omega^1_{22} = -\frac{1}{2}\lambda_m(D_1)\eta^2_1, \quad \omega^1_{33} = -\frac{1}{2r}\lambda_m(D_1)\eta^2_1,$$
$$\omega^1_{12} = \|P_{12}\|\,\|B_{21}\|\eta_1\eta_2, \quad \omega^1_{23} = \xi_1\|B_{21}\|\eta_1\eta_2, \quad \omega^1_{13} = \xi_1\|A_{21}\|\eta_1\eta_1,$$
$$\omega^2_{11} = \lambda_M(A^{\mathrm T}_{22}P_{22} + P_{22}A_{22} + D_2)\eta^2_2 + \eta_1\eta_2\lambda_M(P_{21}A_{12} + A^{\mathrm T}_{12}P^{\mathrm T}_{21}),$$
$$\omega^2_{22} = -\frac{1}{2}\lambda_m(D_2)\eta^2_2, \quad \omega^2_{33} = -\frac{1}{2r}\lambda_m(D_2)\eta^2_2,$$
$$\omega^2_{12} = \|P_{12}\|\,\|B_{12}\|\eta_1\eta_2, \quad \omega^2_{23} = \xi_2\|B_{12}\|\eta_1\eta_2, \quad \omega^2_{13} = \xi_2\|A_{12}\|\eta_1\eta_2,$$
$$\omega^1_{ij} = \omega^1_{ji}, \quad \omega^2_{ij} = \omega^2_{ji}, \quad i,j = 1,2,3, \quad i \neq j.$$

Under some restrictions on the sign-definiteness of matrices $\overline{C}$, $\underline{C}$, and Σ_1, Σ_2 the constructed functional is the Liapunov-Krasovskii functional and applying this functional in Section 4 we shall establish new sufficient conditions for asymptotic stability of the equilibrium state $x = 0$ of quasilinear system. For system (3.37) the proposed method of constructing matrix-valued functional is more efficient, since system of equations (3.38) – (3.39) is integrable in the explicit form. By means of functional $v(\varphi_1, \varphi_2)$ in Section 4 we shall establish sufficient stability conditions for the equilibrium state of system (3.37).

4 Stability Analysis of Quasilinear Delay Systems

We consider an autonomous quasilinear delay system (2.4) with decomposition

$$\begin{aligned}\frac{dx_1}{dt} &= A_{11}x_1(t) + A_{12}x_2(t) + B_{11}x_1(t-r) + B_{12}x_2(t-r) + f_1(x, x_t),\\ \frac{dx_2}{dt} &= A_{21}x_1(t) + A_{22}x_2(t) + B_{21}x_1(t-r) + B_{22}x_2(t-r) + f_2(x, x_t),\end{aligned} \tag{4.1}$$

where $x_i \in R^{n_i}$, $i = 1,2$, $x = (x^{\mathrm T}_1, x^{\mathrm T}_2)^{\mathrm T}$, A_{ij} and B_{ij} are constant matrices of appropriate dimensions.

We make the following assumptions on the functions $f_i(x, x_t)$, $i = 1,2$.

Assumption 1 *The functions $f_i(x, x_t)$, $i = 1,2$, satisfy the following conditions*

(1) *the functions $f_i \in C(R^n \times \mathcal{C}^n,\, R^n)$ for $i = 1,2$;*

(2) *the functions $f_i(0,0) = 0$ iff $x = x_t = 0$;*

(3) *there exist constants $c_{ij}, l_{ij} > 0$, $i,j = 1,2$, such that*

$$|f_i(x, x_t)| \le c_{i1}|x_1(0)| + c_{i2}|x_2(0)| + l_{i1}\|x_1\|_{L_2} + l_{i2}\|x_2\|_{L_2},$$

where $|\cdot|$ is an Euclidean norm in R^{n_i}, $\|\cdot\|_{L_2}$ is the L_2-norm.

For the linear system

$$\begin{aligned}\frac{dx_1}{dt} &= A_{11}x_1(t) + A_{12}x_2(t) + B_{11}x_1(t-r) + B_{12}x_2(t-r),\\ \frac{dx_2}{dt} &= A_{21}x_1(t) + A_{22}x_2(t) + B_{21}x_1(t-r) + B_{22}x_2(t-r),\end{aligned} \tag{4.2}$$

using the results of Section 3 we construct the matrix-valued functional

$$U:\; R^n \times \mathcal{C}^n \to R^{2\times 2}$$

with the elements (3.23), (3.24) and (3.29).

Applying functional $U(\varphi_1, \varphi_2)$ one can establish sufficient stability conditions for solution $x = 0$ of system (4.1). First, we introduce the designations

$$\Delta\Sigma_1 = \begin{pmatrix} 2c_{11}\|P_{11}\| + c_{21}\|P_{12}\| & 0 & \|P_{11}\|l_{11} + \dfrac{1}{2}\|P_{12}\|l_{21} \\ 0 & 0 & 0 \\ \|P_{11}\|l_{11} + \dfrac{1}{2}\|P_{12}\|l_{21} & 0 & 0 \end{pmatrix},$$

$$\Delta\Sigma_2 = \begin{pmatrix} 2c_{22}\|P_{22}\| + c_{12}\|P_{12}\| & 0 & \|P_{22}\|l_{22} + \dfrac{1}{2}\|P_{12}\|l_{12} \\ 0 & 0 & 0 \\ \|P_{22}\|l_{22} + \dfrac{1}{2}\|P_{12}\|l_{12} & 0 & 0 \end{pmatrix},$$

$$\Delta\Sigma_{12} = \begin{pmatrix} 2c_{12}\|P_{11}\| + 2c_{21}\|P_{22}\| + c_{22}\|P_{12}\| + c_{11}\|P_{12}\| & 0 & 2\|P_{22}\|l_{21} + \|P_{12}\|l_{12} \\ 0 & 0 & 0 \\ 2\|P_{11}\|l_{12} + \|P_{12}\|l_{22} & 0 & 0 \end{pmatrix}.$$

Theorem 4.1 *Let system of equations (4.1) be such that*

(1) *there exist solutions of equations (3.25) – (3.27) under initial conditions (3.28) for some* $\eta \in R^2_+,\ \eta > 0$;

(2) *matrices* $\underline{C}$ *and* $\overline{C}$ *in estimate (3.32) are positive definite;*

(3) *matrices* $\Sigma_1 + \Delta\Sigma_1$ *and* $\Sigma_2 + \Delta\Sigma_2$ *are negative definite;*

(4) *inequality*

$$\|\Sigma_{12}\| < \lambda_M(\Sigma_1 + \Delta\Sigma_1)\lambda_M(\Sigma_2 + \Delta\Sigma_2)$$

holds true.

Then the solution $x = 0$ *of system (4.1) is uniformly asymptotically stable.*

Proof Condition (2) of Theorem 4.1 ensures the possibility of constructing the "scalar" functional $v\ R^n \times \mathcal{C}^n \times R^2_+ \to R_+,\ v(\varphi_1, \varphi_2, \eta) = \eta^{\mathrm{T}} U(\varphi_1, \varphi_2)\eta,$ satisfying the conditions of definite positiveness and decrease. The upper right derivative number of the functional $v(\varphi_1, \varphi_2, \eta)$ admits the estimate

$$D^+v(\varphi_1, \varphi_2, \eta)\big|_{(4.1)} \le u_1^{\mathrm{T}}\,(\Sigma_1 + \Delta\Sigma_1)u_1 + 2u_1^{\mathrm{T}}\Sigma_{12}u_2 + u_2^{\mathrm{T}}\,(\Sigma_2 + \Delta\Sigma_2)u_2,$$

where

$$u_1 = (|\varphi_1(0)|,\, |\varphi_1(-r)|,\, \|\varphi_1\|_{L_2})^{\mathrm{T}}, \quad u_2 = (|\varphi_2(0)|,\, |\varphi_2(-r)|,\, \|\varphi_2\|_{L_2})^{\mathrm{T}}.$$

Conditions (3) and (4) ensure definite negativeness of $D^+v(\varphi_1, \varphi_2, \eta)|_{(4.1)}$. Thus, the solution $x = 0$ of system (4.1) is uniformly asymptotically stable and the constructed functional $v(\varphi_1, \varphi_2, \eta)$ is the matrix-valued Liapunov-Krasovskii functional. □

Corollary 4.1 Let system of equations (4.2) be such that

(i) there exist solutions of equations (3.25) – (3.27) under initial conditions (3.28) for some $\eta \in R^2_+$, $\eta > 0$;

(ii) matrices $\underline{C}$ and $\overline{C}$ in estimate (3.32) are positive definite;

(iii) matrices Σ_1 and Σ_2 are negative definite.

Then solution $x = 0$ of system (4.2) is uniformly asymptotically stable.

In the partial case, when $B_{11} = 0$ and $B_{22} = 0$ sufficient conditions of uniform asymptotic stability of solution $x = 0$ are formulated in terms of estimates (3.45) and (3.46) for matrix-valued functional $U(\varphi_1, \varphi_2)$ with the elements

$$w_{11}(\varphi_1) = \varphi_1^{\mathrm{T}}(0)P_{11}\varphi_1(0) + \int_{-r}^{0} k(\theta)\varphi_1^{\mathrm{T}}(\theta)D_1\varphi_1(\theta)\,d\,\theta,$$

$$w_{22}(\varphi_2) = \varphi_2^{\mathrm{T}}(0)P_{22}\varphi_2(0) + \int_{-r}^{0} k(\theta)\varphi_2^{\mathrm{T}}(\theta)D_2\varphi_2(\theta)\,d\,\theta,$$

$$\begin{aligned} w_{12}(\varphi_1, \varphi_2) = \varphi_1^{\mathrm{T}}(0)P_{12}\varphi_2(0) &+ \frac{\eta_1}{\eta_2}\varphi_1^{\mathrm{T}}(0)\int_{-r}^{0} \exp\{A_{11}^{\mathrm{T}}(\theta + r)\}P_{11}B_{12}\varphi_2(\theta)\,d\theta \\ &+ \frac{\eta_2}{\eta_1}\varphi_2^{\mathrm{T}}(0)\int_{-r}^{0} B_{21}^{\mathrm{T}}P_{22}\exp\{A_{22}(\theta + r)\}\varphi_2(\theta)\,d\theta. \end{aligned}$$

Theorem 4.2 *Let system of equations (4.1) be such that*

(1) $B_{11} = 0$, $B_{22} = 0$*;*

(2) *matrices* $\underline{C}$ *and* $\overline{C}$ *in estimates (3.45) are positive definite;*

(3) *matrices* $\Omega_1 + \Delta\Sigma_1$ *and* $\Omega_2 + \Delta\Sigma_2$ *from estimate (3.46) are negative definite;*

(4) *inequality*

$$\|\Sigma_{12}\| < \lambda_M(\Omega_1 + \Delta\Sigma_1)\lambda_M(\Omega_2 + \Delta\Sigma_2)$$

holds true.

Then solution $x = 0$ *of system (4.1) is uniformly asymptotically stable.*

The proof is similar to the proof of Theorem 4.1.

Corollary 4.2 Let system of equations (4.2) be such that

(i) matrices $B_{11} = 0$ and $B_{22} = 0$;

(ii) matrices $\underline{C}$ and $\overline{C}$ in estimate (3.45) are positive definite;

(iii) matrices Ω_1 and Ω_2 in estimate (3.46) are negative definite.

Then solution $x = 0$ of system (4.2) is uniformly asymptotically stable.

5 Application

As applications of results of Section 3.3 we consider oscillations of a harmonic oscillator. The perturbed motion equation of the oscillator is

$$\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + \omega^2 x(t) + cx(t-r) = 0, \tag{5.1}$$

where x is a state variable, ω, c, $\mu > 0$ are constants. Introduce an auxiliary variable $y = \dfrac{dx}{dt}$ and present equation (5.1) as a system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -\omega^2 x(t) - \mu y(t) - cx(t-r). \end{aligned} \tag{5.2}$$

Applying the proposed technique of construction of the Liapunov functionals for system (5.2) we construct a scalar functional $w(\varphi_1, \varphi_2)$ as

$$w(\varphi_1, \varphi_2) = v_{11}(\varphi_1) + 2v_{12}(\varphi_1, \varphi_2) + v_{22}(\varphi_2), \tag{5.3}$$

where

$$v_{11}(\varphi_1) = \gamma^2\varphi_1^2(0) + \gamma^2 d_1 \int_{-r}^{0} \left(1 + \frac{\theta}{2r}\right)\varphi_1^2(\theta)\, d\theta,$$

$$v_{22}(\varphi_2) = \varphi_2^2(0) d_2 \int_{-r}^{0} \left(1 + \frac{\theta}{2r}\right)\varphi_2^2(\theta)\, d\theta,$$

$$v_{12}(\varphi_1, \varphi_2) = 2\frac{\gamma^2 - \omega^2 - ce^{-\mu r}}{\mu}\varphi_1(0)\varphi_2(0) - 2ce^{-\mu r}\varphi_2(0)\int_{-r}^{0} e^{-\mu\theta}\varphi_1(\theta)\, d\theta,$$

and γ, d_1 and d_2 are indefinite positive constants.

Functional (5.3) can be estimated from below by means of the Cauchy–Bunyakovsky inequality

$$\begin{aligned} w(\varphi_1, \varphi_2) &\geq \gamma^2\varphi_1^2(0) + \varphi_2^2(0) - 2\frac{\gamma^2 - \omega^2 - ce^{-\mu r}}{\mu}|\varphi_1(0)|\,|\varphi_2(0)| \\ &+ \frac{\gamma^2 d_1}{2}\|\varphi_1(\theta)\|_{L_2}^2 + \frac{d_2}{2}\|\varphi_2(\theta)\|_{L_2}^2 - 2|c|e^{-\mu r}\sqrt{\frac{e^{2\mu r} - 1}{2\mu}}|\varphi_2(0)|\,\|\varphi_1(\theta)\|_{L_2}. \end{aligned}$$

The derivative of functional (5.3) along solutions of system (5.2) is

$$
\begin{aligned}
D^+w(\varphi_1,\varphi_2)\big|_{(5.2)} &= \Big(d_1\gamma^2 - \frac{2\omega^2(\gamma^2-\omega^2-ce^{-\mu r})}{\mu}\Big)\varphi_1^2(0) \\
&- \frac{d_1\gamma^2}{2}\,\varphi_1^2(-r) - \frac{d_1\gamma^2}{2r}\,\|\varphi_1(\theta)\|^2_{L_2} - \frac{2c(\gamma^2-\omega^2-ce^{-\mu r})}{\mu}\,\varphi_1(0)\varphi_1(-r) \\
&+ 2ce^{-\mu r}\omega^2\varphi_1(0)\int_{-r}^{0} e^{-\mu\theta}\varphi_1(\theta)\,d\theta + 2c^2e^{-\mu r}\varphi_1(-r)\int_{-r}^{0} e^{-\mu\theta}\varphi_1(\theta)\,d\theta \\
&+ \Big(-2\mu + d_2 + \frac{2(\gamma^2-\omega^2-ce^{-\mu r})}{\mu}\Big)\varphi_2^2(0) - \frac{d_2}{2}\,\varphi_2^2(-r) - \frac{d_2}{2r}\,\|\varphi_2(\theta)\|^2_{L_2}.
\end{aligned}
\tag{5.4}
$$

The analysis of (5.4) shows that it is reasonable to take constants $\gamma^2 = \omega^2 + ce^{-\mu r} + \dfrac{\mu^2}{2}$, $d_1 = \dfrac{\mu\omega^2}{2\gamma}$ and $d_2 = \dfrac{\mu}{2}$. Applying the Cauchy-Bunyakovsky inequality once again we estimate derivative (5.4) of functional (5.3)

$$
\begin{aligned}
D^+w(\varphi_1,\varphi_2)\big|_{(5.1)} &\le \Big(d_1\gamma^2 - \frac{2\omega^2(\gamma^2-\omega^2-ce^{-\mu r})}{\mu}\Big)\varphi_1^2(0) \\
&- \frac{d_1\gamma^2}{2}\,\varphi_1^2(-r) - \frac{d_1\gamma^2}{2r}\,\|\varphi_1(\theta)\|^2_{L_2} + \frac{2|c|(\gamma^2-\omega^2-ce^{-\mu r})}{\mu}\,|\varphi_1(0)||\varphi_1(-r)| \\
&+ 2|c|e^{-\mu r}\omega^2\sqrt{\frac{e^{2\mu r}-1}{2\mu}}|\varphi_1(0)|\,\|\varphi_1(\theta)\|_{L_2} \\
&+ 2c^2e^{-\mu r}\sqrt{\frac{e^{2\mu r}-1}{2\mu}}|\varphi_1(-r)|\,\|\varphi_1(\theta)\|^2_{L_2} \\
&+ \Big(-2\mu + d_2 + \frac{2(\gamma^2-\omega^2-ce^{-\mu r})}{\mu}\Big)\varphi_2^2(0) - \frac{d_2}{2}\,\varphi_2^2(-r) - \frac{d_2}{2r}\,\|\varphi_2(\theta)\|^2_{L_2}.
\end{aligned}
$$

Conditions of positive definiteness of functional $w(\varphi_1,\varphi_2)$ and negative definiteness of functional $D^+w(\varphi_1,\varphi_2)|_{(5.2)}$ yield new conditions of asymptotic stability of zero solution of equation (5.2) in the form of the system of inequalities

$$
|c| < \frac{\mu}{2}\sqrt{\frac{\mu}{r(1-e^{-2\mu r})}}, \quad \omega^2 > |c|\sqrt{\frac{24r(1-e^{-2\mu r})+2\mu^3}{\mu^3-4c^2r(1-e^{-2\mu r})}},
$$

$$
\big(2\omega^2 + 2ce^{-\mu r} + \mu^2\big)\big(\mu^2\omega^2 - 2c^2(1-e^{-2\mu r})\big) \ge \frac{\mu^4\omega^2}{2}.
$$

6 Concluding Remarks

It is of interest to apply the proposed approach for a class of neutral functional differential equations with time-varying delay. In [14] the Liapunov functional $V(x(t))$ is used and a condition for asymptotic stability of zero solution of the system under consideration is established.

Another class of equations being of interest for the application of the method are the logic-dynamical hybrid systems given by a set of subsystems which are linear differential-difference equations with constant coefficients and constant delay (see [10]).

An urgent direction of applications is the analysis of the robust stability of nonlinear uncertain neural networks with constant or time-varying delay (see [13]) as well as the problem of robust dynamic parameter-dependent output feedback stabilization under H_∞ performance index for a class of linear time-invariant parameter-dependent systems with multi-time delays in the state vector and in the presence of norm-bounded nonlinear uncertainties (see [8]).

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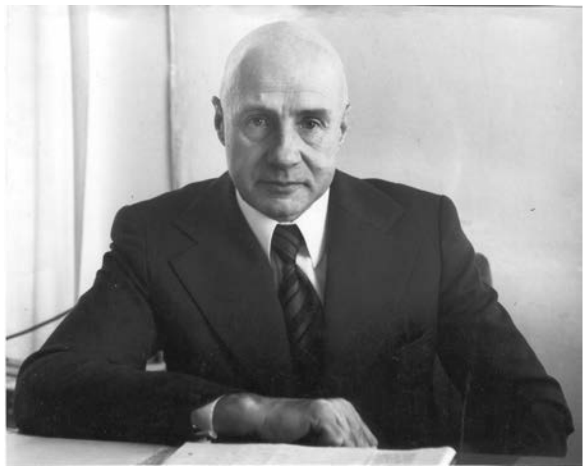
Obituary for Yurii Alexeevich Mitropolskii (1917 – 2008)



The mathematical world is very saddened and shocked to learn about the unexpected death of Yurii Alexeevich Mitropolskii in Kiev, Ukraine on June 14, 2008.

Yurii Alexeevich Mitropolskii was born on January 3, 1917 in the Charnysh's estate located in Kobelyakskiy district of Poltava province.

During the civil war (in 1918), the Charnysh's residence was completely destroyed. It was common for that time. Consequently, the Mitropolskii's moved to Kiev. In 1932, Yurii Alexeevich finished a 7-years school in Kiev followed by his employment at a cannery. In 1938, Yurii Alexeevich graduated from a high school with honors. In the same year, he was accepted at Kiev University in the department of mathematics and physics.

Upon completion of his third year at Kiev University, when on the day of June 22, 1941, the fascist Germany attacked the Soviet Union, Yurii Alexeevich married his university mate Alexandra Ivanovna to live together happily for more than 60 years.

On July 7, 1941 Yurii Alexeevich joined the Soviet Army and was stationed in an armor division in the city of Chuguyev. In October of 1941, according to the decree issued by the Defense Secretary S.K. Timoshenko, all fourth and fifth years college students were eligible to continue their degrees at the corresponding universities, with forthcoming appointments at military academies. Yurii Alexeevich was sent to the city of Kzyl-Orda in Kazakhstan where Kiev University was evacuated to. In March of 1942, Yurii Alexeevich successfully passed all exams and graduated from Kazakh University and then was sent to Ryazan Artillery Academy in the city of Talgar, which he graduated from in March of 1943 in the rank of a lieutenant. He was sent to the Stepnoy battlefield thereafter.

In 1946, after being discharged from the army, Yurii Alexeevich joined the Ukrainian Academy of Sciences in Kiev in the capacity of a Junior Scientist. In 1948 Yurii Alexeevich received his Candidate of Science degree (the Western equivalent of a Ph.D. degree). His thesis was titled “Investigation of resonance phenomena in nonlinear systems with variable frequencies”. In the same year he joined the Institute of Constructive Mechanics of the Ukrainian Academy of Sciences (now S.P. Timoshenko’s Institute of Mechanics of the National Academy of Sciences of Ukraine) in the capacity of a Senior Scientist under the supervision of N.N. Bogoliubov.

In 1951 he was awarded a Doctor of Science degree (the Western and Eastern European equivalent of Habilitation Degree). His thesis was titled “Slow processes in nonlinear oscillatory systems with many degrees of freedom”. Earlier he moved to the Institute of Mathematics of the Ukrainian Academy of Sciences where he was appointed a Senior Scientist. In 1953 Yurii Alexeevich was promoted to the rank of Professor and Department Head in the same Institute. In 1956 he became the Associate Provost of Science of this Institute and in 1958 he was appointed its Director. He remained in this capacity up until 1988. Since1988 he has served as the Honorary Director of the Institute of Mathematics.

In 1958, Yurii Alexeevich was elected the Corresponding Member of the Academy of Sciences of Ukrainian SSR and in 1961 he was elected the Full Member of the Academy of Sciences of the USSR (now the Russian Academy of Sciences), then the most prestigious academic title in the USSR.

During the years of his scientific activity, Yurii Alexeevich Mitropolskii has obtained numerous fundamental results in nonlinear mechanics and differential equations. The results of his prolific research were manifested in more than 700 articles and 50 monographs, of which most essential are “Nonstationary Processes in Nonlinear Oscillating Systems” (1955), “Asymptotic Methods in the Theory of Nonlinear Oscillations” (1964), “Averaging Method in Nonlinear Mechanics” (1971), and “Nonlinear Mechanics. Single-Frequency Oscillations” (1997).

We present an overview of his most significant work. The main directions of his science investigations are the following:

- Development of Asymptotic Methods in Nonlinear Mechanics
- The Development of the Single-Frequency Method
- Contribution to the Method of Integral Manifolds
- The Method of Accelerated Convergence
- The Averaging Method
- Asymptotic Methods and Averaging Method for Distributed Parameter Systems
- Contribution to the Theory of Systems with Delay and Small Parameter
- Development of the Theory of Random Oscillating Processes
- Contribution to the Theory of Decomposition of Systems

Since 1958, Mitropolskii has focused his attention on the development of the Institute of Mathematics of Academy of Science of the USSR. He initiated new departments setting up to facilitate research in the areas of algebra, probability theory, real and functional analysis, and mechanics of special systems.

During this period of time, the post-graduate enrollment was substantially expanded. As the result of Mitropolskii’s efforts, the Institute produced about 500 candidates of science and more than 80 doctors of science for their further employments at national universities and research labs in Ukraine, Russia, and other countries. As a consequences of Mitropolskii's colossal scholarly activity, the Institute of Mathematics of Academy of Science of Ukranian SSR has become the leading scientific center of mathematical research in Ukraine.

Mitropolskii began his pedagogical activity in 1948 at Kiev university to extend it up to 1989. Mitropolskii himself supervised and directed 100 Ph.D. and 25 Habilitation theses in physical and mathematical sciences.

From 1961 until 1992 Mitropolskii had been the Head of the Department of mathematics, mechanics and cybernetics at Academy of Sciences of Ukrainian SSR. In 1992 Mitropolskii was appointed the director of the International mathematical center of National Academy of Science of Ukraine and Counselor of Presidium of National Academy of Science of Ukraine. This position he had held until his death.

Mitropolskii has been much involved in editorial work. Since 1967, Mitropolskii has been the Editor-in Chief of the "Ukrainian Mathematical Journal" whose English translation is regularly published in the US. Since 1961, he has been an editorial board member of three Russian and three international journals. Mitropolskii was among main contributors to a 12-volume selected works by N.N. Bogoliubov in the area of mathematics and nonlinear mechanics.

The first international talk Mitropolskii gave in 1956 at the International congress of mathematicians in Bucharest, Romania. Since 1958, he has been an invited speaker to the International Mathematical Congresses held in Edinburgh, Scotland (1958), Stockholm Sweden (1962), Moscow, Russia (1966), Niece, France (1970), Vancouver, Canada (1974), Warsaw, Poland (1983), Berkeley, USA (1986), and Kyoto, Japan (1990). A series of lectures and talks on individual problems in nonlinear mechanics was given by Mitropolskii at various universities in the USA, China, Vietnam, Czechoslovakia, Poland, Mexico, Canada, Italy, and Yugoslavia and numerous international conferences.

His active cooperation over the past two decades with Vietnamese scientists in the area of nonlinear mechanics and theory of differential equations is worth mentioning.

Mitropolskii has been one of the most celebrated scientists who has ever lived in Ukraine and Russia. Consequently, his research, scholarly, pedagogical activities and public service have been highly revered. He was awarded by almost all known highest and most prestigious prizes ever given to a Soviet citizen. Here is the list of some of them:

Hero of the Socialist Labor; Honored Activist of Science of UkrSSR; Lenin Prize Laureate; State Prize Laureate of Ukraine; Federal Prize Laureate of Soviet Union; Lyapunov Golden Medal; Certificate of the Soviet Supreme Presidium; Certificate of Presidium of Supreme Soviet of UkrSSR; Lenin Golden Medal; Two Red Star Orders; October Revolution Medal; Labor Red Banner Medal; Second-Degree Great Patriotic War Medal; The Fifth Degree Yaroslav Mudryi Order; Bogdan Hmelnitskiy Medal;

N.M. Krylov, N.N. Bogoliubov and M.A. Lavrentiev prizes of Presidium of the Academy of Sciences of Ukrainian SSR.

Also, outside Ukraine and Russia, Mitropolskii has been treated with a highest honor. In 1971, he was elected the foreign member of Bologna Academy of Sciences (Italy) and awarded with a Silver Medal of Czechoslovak Academy of Sciences "For Achievements in Science and Deeds for the Mankind". The government of Vietnam awarded him with the "Friendship" Medals in 1987 and 2001.

Editorial Board of the Journal of Nonlinear Dynamics and Systems Theory express deep condolences to all Mitropolskiis relatives and loved ones. He will be missed by his friends, collaborators, and the entire mathematical community worldwide.

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