Nonlinear Dynamics and Systems Theory, **5**(2) (2005) 157–167

Stability of Dynamical Systems in Metric Space

A.A Martynyuk*

Stability of Processes Department, S.P.Timoshnko Institute of Mechanics,
National Academy of Sciences of Ukraine,
03057, Nesterov str., 3, 03057, Kiev-57, Ukraine

Received: October 21, 2004; Revised: March 15, 2005

Abstract: In the paper a new approach is developed for stability analysis of motions of dynamical systems defined on metric space using matrix-valued preserving mappings. These results are applicable to a much larger class of systems then existing results, including dynamical systems that cannot be determined by the usual classical equations and inequalities. We apply our results in the stability analysis of hybrid systems in general and two-component hybrid systems.

Keywords: *Dynamical system; metric space; hybrid system; asymptotic stability; stability matrix-valued preserving mapping.*

Mathematics Subject Classification (2000): 34G20, 35B35, 37K45, 37L15, 93A15, 93D30.

1 Introduction

This paper presents an approach to stability analysis of dynamical systems determined in metric space. The method of analysis of invariant sets of dynamical systems was proposed by Zubov [11] on the basis of generalized direct Liapunov method. In our approach a generalized comparison principle is used together with the idea of multicomponent mapping (cf. matrix-valued Liapunov functions [5, 6]).

In the present paper, we first developed a matrix-valued preserving mapping for stability analysis of general dynamical systems defined on metric space. To accomplish this, we utilize, as in our earlier work (see [7]), stability preserving matrix-valued mappings. We use the above results to establish the principal Lyapunov theorems for dynamical systems on metric space. Finally, we analyze a class of hybrid systems, using some of these results with particular application to two-component hybrid system.

*Corresponding author: anmart@stability.kiev.ua

2 Basic Concepts and Definitions

Let X be a set of elements (no matter of what nature) and a measure $\rho(x,y)$ be defined for $x,\, y \in X$. The Definitions 2.1 − 2.6 presented here follow in the spirit of the works [1, 2, 9, 11] even if some of the formulations are different.

Definition 2.1 (X,ρ) is a metric space if the following conditions are fulfilled for any $x, y, z \in X$:

(1) $\rho(x,y) \geq 0$,

(2) $\rho(x,y) = 0 \Leftrightarrow x = y$,

(3) $\rho(x,y) = \rho(y,x)$,

(4) $\rho(x,y) \leq \rho(x,z) + \rho(y,z)$,

and, additionally, for any $X_0 \subseteq X$, $\rho(x,X_0) = \inf\limits_{y\in X_0} \rho(x,y)$.

Definition 2.2 A metric space (T,ρ) is called a temporal space if:

(1) T is completely ordered by the ordering "$<$";

(2) T has a minimum element $t_{\min} \in T$, i.e. $t_{\min} < t$ for any $t \in T$, such that $t \neq t_{\min}$;

(3) for any $t_1, t_2, t_3 \in T$ such that $t_1 < t_2 < t_3$ it holds that

$$\rho(t_1,t_3) = \rho(t_1,t_2) + \rho(t_2,t_3);$$

(4) T is unbounded from above; i.e., for any $M > 0$, there exists $t \in T$ such that $\rho(t, t_{\min}) > M$.

Definition 2.3 Let (X,ρ) be a metric space with a subset $A \subseteq X$ and let (T,ρ) be a temporal space with subset $T \subseteq R_+$. A mapping $p(\cdot, a, \tau_0)\colon T_{a,\tau_0} \to X$ is called *a motion* if $p(\tau_0, a, \tau_0) = a$, where $a \in A$, $\tau_0 \in T$ and $T_{a,\tau_0} = [\tau_0, \tau_1) \cap T$ for $\tau_1 > \tau_0$, with τ_1 being a finite value or infinity.

Definition 2.4 Let $T_{a,\tau_0} \times \{a\} \times \{\tau_0\} \to X$ denote the set of mappings of $T_{a,\tau_0} \times \{a\} \times \{\tau_0\}$ into X, $\Lambda = \bigcup_{(a,\tau_0)\in A\times T} (T_{a,\tau_0} \times \{a\} \times \{\tau_0\} \to X)$ and S be a family of motions; i.e.,

$$S \subseteq \{p(\,\cdot\,, a, \tau_0) \in \Lambda\colon\; p(\tau_0, a, \tau_0) = a\}.$$

Then the four-tuple (T, X, A, S) of sets and spaces is called *a dynamical system*.

Note that Definition 2.4 possesses some generality. Specifically,

(i) if X is a normed linear space and every motion $p(\tau, a, \tau_0)$ is assumed to be continuous with respect to τ, a and τ_0, then Definition 2.4 corresponds to the concept of a family of motions in Hahn [3];

(ii) under some additional conditions imposed on $p(\tau, a, \tau_0)$ (see [11], pp.183–184), Definition 2.4 reduces to the concept of a general system introduced by Zubov.

In what follows, we consider dynamical systems satisfying the standard semigroup property

$$p(\tau_2,\, p(\tau_1, a, \tau_0)) = p(\tau_2 + \tau_1,\, a, \tau_0)$$

for all $a \in A$ and any $\tau_1, \tau_2 \in R_+$.

Definition 2.5 A dynamical system (R_+, X, A, S) is called *continuous* if any of its motions $p \in S$ is continuous; i.e., any mapping $p(\,\cdot\,, a, \tau_0)\colon\ T_{a,\tau_0} \to X$ is continuous.

Let (X_1, ρ_1) and (X_2, ρ_2) be metric spaces, and let (R_+, X_1, A_1, S_1) be a continuous dynamical system. We assume that the space X_1 is a Descartes product of spaces X_{11}, X_{12}, $\dots$, X_{1m}, on which the multicomponent mapping (see [7])

$$U(t, x)\colon\ T \times X_{11} \times X_{12} \times \ldots \times X_{1m} \to X_2 \tag{1}$$

is acting.

It is assumed that the mapping $U\colon\ R_+ \times X_{11} \times X_{12} \times \ldots \times X_{1m} \to X_2$ has the following properties: for any motion $p(\,\cdot\,, a, t_0) \in S_1$, the function $q(\,\cdot\,, b, t_0) = U(\,\cdot,\ p(\,\cdot,\, a, t_0),\,)$ with initial value $b = U(t_0, a)$ is another motion for which $T_{a,t_0} = T_{b,t_0}$, $b \in A_2 \subset X_2$.

Let S_2 denote the set of motions q determined by initial values $a \in A_1$ and $t_0 \in R_+$. Then (R_+, X_2, A_2, S_2) is a continuous dynamical system.

The mapping given by (1) induces a mapping of S_1 into S_2, denoted by $\mathfrak{M}$; i.e., $S_2 = \mathfrak{M}(S_1)$. Moreover, we denote by $M_1 \subset A_1$ and $M_2 \subset A_2$ some sets invariant under S_1 and S_2, respectively. The set M_2 is then defined by the formula

$$\begin{gathered} M_2 = U(R_+ \times M_1) = \{x_2 \in X_2\colon\ x_2 = U(t', x_1) \\ \text{for some } x_1 \in M_1 \text{ and } t' \in R_+,\}. \end{gathered} \tag{2}$$

In what follows, we consider continuous dynamical systems (R_+, X_1, A_1, S_1) and (R_+, X_2, A_2, S_2) with invariant sets $M_1 \subset A_1$ and $M_2 \subset A_2$, respectively.

Definition 2.6 Multicomponent mapping (1)

$$U\colon\ R_+ \times X_{11} \times X_{12} \times, \ldots, \times X_{1m} \to X_2 \tag{3}$$

preserves some type of stability of a continuous dynamical system if the sets

$$\begin{gathered} S_2 = \mathfrak{M}(S_1) \triangleq \{q(\,\cdot\,, b, t_0)\colon\ q(t, b, t_0) = U(t,\, p(t, a, t_0)), \\ p(\,\cdot\,, a, t_0) \in S_1, \quad \eta \in R^m, \quad b = U(t_0, a), \\ T_{b,t_0} = T_{a,t_0}, \quad a \in A_1, \quad t_0 \in R_+\} \end{gathered} \tag{4}$$

and M_2 (see formula (2)) satisfy the following conditions:

(1) the invariance of (S_1, M_1) is equivalent to the invariance of (S_2, M_2);
(2) some type of stability of (S_1, M_1) is equivalent to the same type of stability of (S_2, M_2).

3 Sufficient Conditions for Stability of Dynamical System

Note that the mapping U induces a mapping $\mathfrak{M}\colon\ S_1 \to S_2$, that preserves some types of stability of (S_1, M_1) and $(S_2,\, U(R_+ \times M_1))$.

Theorem 3.1 *Let a dynamical system* (R_+, X_1, A_1, S_1) *be assigned a comparison system* (R_+, X_2, A_2, S_2) *by means of a multicomponent mapping* $U(t,p)\colon R_+ \times X_1 \to X_2$. *Suppose that there exist closed sets* $M_i \subset A_i$, $i = 1, 2$, *and following conditions are fulfilled:*

(1) *for* $\mathfrak{M}(S_1)$ *and* S_2, $\mathfrak{M}(S_1) = S_2$;

(2) *there exist constant* $m \times m$ *matrix* A_i, $i = 1, 2$, *and comparison functions* $\psi_1, \psi_2 \in K$ *such that*

$$\psi_1^{\mathrm{T}} A_1 \psi_1 \le \rho_2(U(t,p),\, M_2) \le \psi_2^{\mathrm{T}} A_2 \psi_2 \tag{5}$$

for all $p \in X_1$ *and* $t \in R_+$, *where*

$$\psi_1 = (\psi_{11}(\rho_1(p, M_1)),\, \dots,\, \psi_{1m}(\rho_1(p, M_1)))^{\mathrm{T}},$$
$$\psi_2 = (\psi_{21}(\rho_1(p, M_1)),\, \dots,\, \psi_{2m}(\rho_1(p, M_1)))^{\mathrm{T}}.$$

Here, ρ_1 *and* ρ_2 *are metrics defined on* X_1 *and* X_2, *respectively.*

If the matrices A_i, $i = 1, 2$, *are positive definite, then the following is true:*

(1) *the invariance of* (S_2, M_2) *implies the invariance of* (S_1, M_1);

(2) *the stability, uniform stability, asymptotic stability, or uniform asymptotic stability of* (S_2, M_2) *implies the respective type of stability of* (S_1, M_1);

(3) *if in estimate (5)* $\psi_1^{\mathrm{T}} A_1 \psi_1 = a(\rho_1(p, M_1))^b$, *where* $a > 0$ *and* $b > 0$, *then the exponential stability of* (S_2, M_2) *implies the exponential stability of* (S_1, M_1).

Proof of item (1) Let (S_2, M_2) be an invariant pair. Then, for any $a \in M_1$ and any motion $p(\,\cdot\,; a, t_0) \in S_1$, we find that $q(\,\cdot\,; b, t_0) = U(t,\, p(\,\cdot\,; a, t_0)) \in S_2$, where $b = U(t_0, a)$. This follows from condition (1) in Theorem 3.1 and from the definition of $\mathfrak{M}(S_1)$ by formula (4). Moreover, the invariance of (S_2, M_2) implies that $q(t; b, t_0) = U(t,\, p(t; a, t_0)) \in M_2$ for all $t \in T_{b,t_0} = T_{a,t_0}$. Since M_1 and M_2 are closed and the matrices A_1 and A_2 are positive definite and satisfy (5), we conclude that $p(t; a, t_0) \in M_1$ for all $t \in T_{a,t_0}$. This implies the invariance of (S_1, M_1).

Proof of item (2) Assume that (S_2, M_2) is stable. Then, by the definition of stability, for any $\varepsilon_2 > 0$ and $t_0 \in R_+$, there exists $\delta_2 = \delta_2(t_0, \varepsilon_2) > 0$ such that $\rho_2(q(t; b, t_0),\, M_2) < \varepsilon_2$ for all $q(\,\cdot\,; b, t_0) \in S_2$ and all $t \in T_{b,t_0}$ whenever $\rho_2(b, M_2) < \delta_2(t_0, \varepsilon_2)$. Estimates (5) can be transformed into

$$\lambda_m(A_1)\widetilde{\psi}_1(\rho_1(p, M_1)) \le \rho_2(U(t,p),\, M_2) \le \lambda_M(A_2)\widetilde{\psi}_2(\rho_1(p, M_1)). \tag{6}$$

Here $\lambda_m(A_1) > 0$ and $\lambda_M(A_2) > 0$ are the minimum and maximum eigenvalues of the positive definite matrices A_1 and A_2, and $\widetilde{\psi}_1, \widetilde{\psi}_2 \in K$ are such that

$$\psi_1^{\mathrm{T}}(\rho_1(p, M_1))\psi_1(\rho_1(p, M_1)) \ge \widetilde{\psi}_1(\rho_1(p, M_1))$$

and

$$\psi_2^{\mathrm{T}}(\rho_1(p, M_1))\psi_2(\rho_1(p, M_1)) \ge \widetilde{\psi}_2(\rho_1(p, M_1)).$$

Since (S_2, M_2) is stable, for any $\varepsilon > 0$ and any $t_0 \in R_+$, we choose $\varepsilon_2 = \lambda_m(A_1)\widetilde{\psi}_1(\varepsilon)$ and $\delta_1 = \lambda_M^{-1}(A_2)\widetilde{\psi}_2^{-1}(\delta_2)$. Assuming that $\rho_1(a, M_1) < \delta_1$ and taking into account (6), we obtain

$$\begin{aligned}\rho_2(b, M_2) &\le \lambda_M(A_2)\widetilde{\psi}_2(\rho_1(a, M_1)) < \lambda_M(A_2)\widetilde{\psi}_2(\delta_1)\\ &= \lambda_M(A_2)\widetilde{\psi}_2(\lambda_M^{-1}(A_2)\widetilde{\psi}_2^{-1}(\delta_2)) = \delta_2.\end{aligned}$$

It follows that, for all motions $q(\,\cdot\,; b, t_0) \in S_2$, the estimate $\rho_2(q(t; b, t_0), M_2) < \varepsilon_2$ holds for all $t \in T_{b,t_0}$. Returning to estimates (6), we find that, for all $p(\,\cdot\,; a, t_0) \in S_1$ and all $t \in T_{a,t_0} = T_{b,t_0}$, where $b = U(t_0, a)$, we have

$$\begin{aligned}\rho_1(p(t; a, t_0), M_1) &\le \lambda_m^{-1}(A_1)\widetilde{\psi}_1^{-1}(\rho_2(V(p(t; a, t_0)), M_2))\\ &\le \lambda_m^{-1}(A_1)\widetilde{\psi}_1^{-1}(\rho_2(q(t; b, t_0), M_2)) \le \lambda_m^{-1}(A_1)\widetilde{\psi}_1^{-1}(\lambda_m(A_1)\widetilde{\psi}_1(\varepsilon)) = \varepsilon,\end{aligned}$$

whenever $\rho_1(a, M_1) < \delta_1$. It follows that (S_1, M_1) is stable.

It is well known that a system motion is asymptotically stable if it is stable and attracting. Assume that (S_2, M_2) is attracting. Then, for any $t_0 \in R_+$ there exists $\Delta_2 = \Delta_2(t_0) > 0$ such that, for all $q(\,\cdot\,; b, t_0) \in S_2$, the limit relation

$$\lim_{t\to\infty} \rho_2(q(t; b, t_0), M_2) = 0,$$

holds true whenever $\rho_2(b, M_2) < \Delta_2$. In other words, for any $\varepsilon_2 > 0$, there exists $\tau = \tau(\varepsilon_2, t_0, q) > 0$ with $q = q(\,\cdot\,; b, t_0) \in S_2$ such that $\rho_2(q(t; b, t_0), M_2) < \varepsilon_2$ for all $t \in T_{b,t_0+\tau}$, whenever $\rho_2(b, M_2) < \Delta_2$. According to condition (1) in Theorem 3.1, for any motion $p(\,\cdot\,; a, t_0) \in S_1$, we set $b = U(t_0, a)$. Then $q(\,\cdot\,; b, t_0) = U(p(\,\cdot\,; a, t_0)) \in S_2$. Furthermore, for any $\varepsilon_1 > 0$, we choose $\varepsilon_2 = \lambda_m(A_1)\widetilde{\psi}_1(\varepsilon_1)$ and set $\Delta_1 = \lambda_M^{-1}(A_2)\widetilde{\psi}_2^{-1}(A_2)$. For any motion $p(\,\cdot\,; a, t_0) \in S_1$, we then have

$$\rho_2(b, M_2) \le \lambda_M(A_2)\widetilde{\psi}_2(\rho_1(a, M_1)) < \lambda_M(A_2)\widetilde{\psi}_2(\Delta_1) = \Delta_2$$

whenever $\rho_1(a, M_1) < \Delta_1$ and $t \in T_{a,t_0+\tau} = T_{b,t_0+\tau}$. Hence, $\rho_2(q(t; a, t_0), M_2) < \varepsilon_2 = \lambda_m(A_1)\widetilde{\psi}_1(\varepsilon_1)$ for all $t \in T_{a,t_0+\tau}$. Returning to estimate (2), we find that

$$\rho_1(p(t; a, t_0), M_1) \le \lambda_m^{-1}(A_1)\widetilde{\psi}_1^{-1}(\rho_2(q(t; a, t_0), M_2)) < \lambda_m^{-1}(A_1)\widetilde{\psi}_1^{-1}(\varepsilon_1),$$

i.e., (S_1, M_1) is an attractive pair. Thus, if (S_2, M_2) is asymptotically stable, then (S_1, M_1) is asymptotically stable as well.

The statements on uniform stability and uniform asymptotic stability are proved following the same scheme, but δ_2 and Δ_2 are chosen to be independent of $t_0 \in R_+$.

Let us prove statement (3) of the theorem. Assume that (S_2, M_2) is exponentially stable. Then there exists $\alpha_2 > 0$ and, for any $\varepsilon_2 > 0$, there exists $\delta_2 = \delta_2(\varepsilon_2) > 0$ such that for any motion $q(\,\cdot\,; b, t_0) \in S_2$ and all $t \in T_{b,t_0}$

$$\rho_2(q(t; b, t_0), M_2) < \varepsilon_2 e^{-\alpha_2(t-t_0)}$$

whenever $\rho_2(b, M_2) < \delta_2$. According to condition (1) in Theorem 3.1, for any motion $p(\,\cdot\,; a, t_0) \in S_1$, there exists a motion $q(\,\cdot\,; b, t_0) = U(p(\,\cdot\,; a, t_0)) \in S_2$, where $b =$

$U(t_0, a)$. Furthermore, for any $\varepsilon_1 > 0$, we choose $\varepsilon_2 = a\varepsilon_1^b$. Let $\alpha_1 = \alpha_2/b$ and $\delta_1 = \lambda_M^{-1}(A_2)\psi_2^{-1}(\delta_2)$. For $p(t; a, t_0) \in M_1$ with $\rho_1(a, M_1) < \delta_1$, in view of (6), we obtain

$$\rho_2(b, M_2) \le \lambda_M(A_2)\widetilde{\psi}_2(\rho_1(a, M_1)) < \lambda_M(A_2)\widetilde{\psi}_2(\delta_1) = \delta_2.$$

Consequently,

$$\rho_2(q(t; b, t_0), M_2) < \varepsilon_2 e^{-\alpha_2(t-t_0)}$$

for all $t \in T_{b,t_0}$.

According to the hypothesis of Theorem 3.1, we have to set

$$\psi_1^{\mathrm{T}} A_1 \psi_1 = a(\rho_1(p, M_1))^b$$

in (1.6.6). It is easy to see that

$$\rho_1(p(t; a, t_0), M_1) < \left(\frac{\varepsilon_2}{a}\right)^{1/b} e^{-\frac{\alpha_2}{b}(t-t_0)} = \varepsilon_1 e^{-\alpha_1(t-t_0)}$$

for all $t \in T_{a,t_0}$. Thus, (S_1, M_1) is exponentially stable.

4 Stability Analysis of Hybrid System

Many physical and technical problems of real world are modelled by mixed systems of equations and correlations. For example, in motion control theory the feedback consists of several interconnected blocks. These blocks are described by equations of different types. Such systems are called hybrid (see [9]). Under certain assumptions real hybrid system σ can correspond to the dynamical system (T, X, A, S) in metric space.

Assume that (X, ρ) and (X_i, ρ_i), $i = 1, 2, \dots, m$, are metric spaces. Let $X = X_1 \times X_2 \times \ldots \times X_m$ and there exist constants $a_1, a_2 > 0$ such that

$$a_1\rho(x, y) \le \sum_{i=1}^{m} \rho_i(x_i, y_i) \le a_2\rho(x, y) \tag{7}$$

for all $x, y \in X$, where $x = (x_1, \dots, x_m)^{\mathrm{T}}$, $y = (y_1, \dots, y_m)^{\mathrm{T}}$, $x_i \in X_i$, $y_i \in X_i$, $i = 1, 2, \dots, m$. Further on we will assume that

$$\rho(x, y) = \sum_{i=1}^{m} \rho_i(x_i, y_i). \tag{8}$$

Definition 4.1 (cf. [9]) Dynamical system (T, X, A, S) is *hybrid*, if its metric space (X, ρ) consists of metric spaces (X_i, ρ_i), $i = 1, 2, \dots, m$, where X_i are nontrivial unsplitted with metrics $\rho_i(x_i, y_i)$, and if there exist at least two metric spaces X_i and X_j, $1 \le i \ne j \le m$, which are not isometric.

The proposition below is necessary when the multicomponent mapping is made by matrix-valued functional.

Proposition 4.1 *Let multicomponent mapping* $U(t,x)\colon T\times X\to X_2$ *be performed by matrix-valued functional* $U(t,x)=[v_{ij}(t,x)]$, $i,j=1,2,\dots,m$, *for the elements of which:*

(a) $v_{ii}\in C(R_+\times X,\,R_+)$, $i=1,2,\dots,m$, $v_{ij}\in C(R_+\times X,\,R)$ *for all* $i\neq j$ *and for all* $x\in X$ *and* $t\in R_+$*;*

(b) *there exist comparison functions* $\varphi_{i1},\varphi_{i2}$ *of class* K*, positive constants* $\underline{c}_{ii}>0$, $\overline{c}_{ii}>0$ *and arbitrary constants* $\underline{c}_{ij}\in R$, $\overline{c}_{ij}\in R$ *for* $i\neq j$ *such that*

$$\begin{gathered}\underline{c}_{ii}\varphi_{i1}^2(\rho_i(x_i,M_i))\le v_{ii}(t,x)\le\overline{c}_{ii}\varphi_{i2}^2(\rho_i(x_i,M_i)),\\ \underline{c}_{ij}\varphi_{i1}(\rho_i(x_i,M_i))\varphi_{j1}(\rho_j(x_j,M_j))\le v_{ij}(t,x)\\ \le\overline{c}_{ij}\varphi_{i2}(\rho_i(x_i,M_i))\varphi_{j2}(\rho_j(x_j,M_j))\end{gathered}\tag{9}$$

for all $x_i\in X_i$, $x\in X$ *and* $t\in R_+$.

Then for the functional

$$v(t,x,\eta)=\eta^{\mathrm T}U(t,x)\eta,\quad \eta\in R_+^m,\quad \eta_i>0,$$

the bilateral inequality

$$\begin{gathered}u_1^{\mathrm T}(\rho(x,M))H^{\mathrm T}\underline{C}Hu_1(\rho(x,M))\le v(t,x,\eta)\\ \le u_2^{\mathrm T}(\rho(x,M))H^{\mathrm T}\underline{C}Hu_2(\rho(x,M))\end{gathered}\tag{10}$$

holds for all $x\in X$ *and* $t\in R_+$, *where*

$$\begin{gathered}H=\operatorname{diag}(\eta_1,\eta_2,\dots,\eta_m),\\ \underline{C}=[\underline{c}_{ij}],\quad \overline{C}=[\overline{c}_{ij}],\quad i,j=1,2,\dots,m,\\ u_1(\cdot)=(\varphi_{i1}(\rho_1(x_1,M_1)),\,\dots,\,\varphi_{m1}(\rho_m(x_m,M_m)))^{\mathrm T},\\ u_2(\cdot)=(\varphi_{i2}(\rho_1(x_1,M_1)),\,\dots,\,\varphi_{m2}(\rho_m(x_m,M_m)))^{\mathrm T}.\end{gathered}$$

Proof Estimate (10) is obtained by direct substitution by estimates (b) of Proposition 4.1 in the expression

$$v(t,x,\eta)=\sum_{i=1}^m\sum_{j=1}^m v_{ij}(t,x)\eta_i\eta_j.$$

Theorem 4.1 *Assume that behaviour of the hybrid system* Σ *is correctly described by the dynamical system* (T,X,A,S), *where* $T=R_+$, $X=X_1\times\ldots\times X_m$ *and* X_i *are subspaces with metrics* ρ_i, $i=1,2,\dots,m$. *Let* $M_i\subset X_i$ *and* $M=M_1\times M_2\times\ldots\times M_m$ *be an invariant set. If*

(1) *there exist functionals* $v_{ij}(t,x)$ *mentioned in Proposition 4.1;*

(2) *given functionals* $v_{ij}(t,x)$ *and a vector* $\eta\in R_+^m$, $\eta>0$, *there exist bounded for all* $x\in X$ *functions* $\Phi_{ij}(x,\eta)$, $i,j=1,2,\dots,m$, *and comparison functions* φ_{i3} *of class* K *such that*

$$D^+v(t,x,\eta)|_{(S)}\le u_3^{\mathrm T}\Phi(x,\eta)u_3$$

on system of motions S *for all* $x\in X$ *and* $t\in R_+$, *where*

$$u_3(\rho(x,M))=(\varphi_{13}(\rho_1(x_1,M_1)),\,\dots,\,\varphi_{m3}(\rho_m(x_m,M_m)))^{\mathrm T}.$$

Then

(a) *If matrices* $B_1 = H^{\mathrm{T}}\underline{C}H,\ B_2 = H^{\mathrm{T}}\overline{C}H$ *are positive definite and constant* $m\times m$ *matrix* $\overline{\Phi} \geq \frac{1}{2}\,(\Phi^{\mathrm{T}}(x,\eta) + \Phi(x,\eta))$ *for all* $x \in X$ *is negative semidefinite, then the couple* (S, M) *is uniformly stable.*

(b) *If matrices* B_1 *and* B_2 *are positive definite and matrix* $\overline{\Phi}$ *is negative definite, then the couple* (S, M) *is uniformly asymptotically stable.*

(c) *If matrices* B_1 *and* B_2 *are positive definite, matrix* $\overline{\Phi}$ *is negative semidefinite, the set* M *is bounded and the comparison functions* $\varphi_{i1}, \varphi_{i2} \in KR$ *class* $i = 1, 2, \ldots, m$, *then the family of motion* S *is uniformly bounded.*

(d) *If in condition (c) the matrix* $\overline{\Phi}$ *is negative definite, then the family of motions* S *is uniform-ultimately bounded and the couple* (S, M) *is uniformly asymptotically stable in the whole.*

(e) *If there exist constants* $a_1,\ a_2,\ b,\ c$ *such that*

$$a_1 r^b \leq u_1^{\mathrm{T}}(\rho(x,M))H^{\mathrm{T}}\underline{C}u_1(\rho(x,M)),$$
$$u_2^{\mathrm{T}}(\rho(x,M))H^{\mathrm{T}}\overline{C}u_2(\rho(x,M)) \leq a_2 r^b,$$
$$\varphi_3^{\mathrm{T}}\overline{\Phi}\varphi_3 \geq c r^b$$

for all $r \in R_+,$ *then the couple* (S, M) *is exponentially stable in the whole.*

Proof Let us prove statement (a) of Theorem 4.1. Under condition (1) of Theorem 4.1 the functional $v(t,x,\eta)$ is positive definite and decreascent because matrices B_1 and B_2 are positive definite. Under condition (2) of Theorem 4.1 the functional $D^+v(t,x,\eta)$ on the system of motions S is negative semidefinite due to restrictions on matrix $\overline{\Phi}$. In this case the functional $v(t,x,\eta)$ is nonincreasing for all $t \geq 0$ along the system of motions S. Further, given $\varepsilon > 0$, we compute $\lambda = \inf\limits_{t\geq 0} v(t,x,\alpha)$ for $\rho(x,M) = \varepsilon$. Because of estimate (9) we can find by value λ the value $\delta > 0$ such that for $\rho(x,M) < \delta$ the estimate $v(t,x,\alpha) < \lambda$ holds for all $t \geq 0$. Now we show that the obtained value $\delta > 0$ corresponds to the given $\varepsilon > 0$, i.e. for $\rho(x,M) < \delta$ the inequality

$$\rho(q(t;a,t_0),\ M) < \varepsilon$$

holds for all $t \geq 0$. Assume on the contrary, let there exist a motion $q(t;a,t_0) \in S$ such that for some value $t^* \in R_+$ the inequality $\rho(q(t^*;a,t_0),\ M) = \varepsilon$ takes place. Then we get

$$v(t,\ q(t^*;a,t_0),\alpha) \geq \lambda,$$

but due to condition (a) of Theorem 4.1 the functional $v(t,x,\alpha)$ is nonincreascent along the system of motions S. Therefore

$$v(t,\ q(t;a,t_0),\alpha) \leq v(t,x,\alpha) < \lambda$$

for any $q(t;a,t_0) \in S$.

The contradiction obtained shows that the system of motions S of the hybrid system Σ is uniformly (S, M) stable.

The proof of statements (b) – (e) of Theorem 4.1 is similar to that of statement (a) following the Liapunov (ε,δ)-technique.

5 Stability Analysis of Two-Component Systems

We consider a hybrid two-component system [4]

$$\frac{dx}{dt} = X(t, x(t)) + g_1(t, z, x(t), w(t, z)), \quad x(t_0) = x_0, \tag{11}$$

$$\frac{\partial w}{\partial t} = L(t, x, \partial/\partial z)w + g_2(t, z, x(t), w(t, z)), \tag{12}$$

where

$$w(t_0, z) = w^0(z), \quad M(t, z, \partial/\partial z)w|_{\partial\Omega} = w^1(t, s), \quad s \in \partial\Omega, \quad \Omega \subset R^k,$$
$$X\colon\ T_0 \times U \to R^n, \quad L\colon\ B_1 \to B_2, \quad M\colon\ B_1 \to B_3, \quad w^0 \in B_4,$$

L, M are some differential operators and $B_1, \dots, B_4$ are Banach spaces.

A hybrid system (11) and (12) consists of the independent subsystems

$$\frac{dx}{dt} = X(t, x(t)), \tag{13}$$

$$\frac{\partial w}{\partial t} = L(t, z, \partial/\partial z)w \tag{14}$$

and interconnection functions between them

$$g_1 = g_1(t, z, x, w)\colon\ T_0 \times \Omega \times H \times Q \to R^n,$$
$$g_2 = g_2(t, z, x, w)\colon\ T_0 \times \Omega \times H \times Q \to R^m.$$

Let us introduce the assumptions on subsystems (13), (14) and interconnection functions between them.

Assumption 5.1 There exist functions $v_{ij} \in C(R_+ \times H \times Q, R)$, $i, j = 1, 2$, $v_{ij}(t, x, w)$ is locally Lipschitzian in x and w, functions of comparison $\varphi_i, \psi_i \in K$, $i = 1, 2$, and positive constants $\underline{\alpha}_{ii}, \overline{\alpha}_{ii} > 0$, $i = 1, 2$, and arbitrary constants $\underline{\alpha}_{12}$, $\overline{\alpha}_{12}$ such that

$$\underline{\alpha}_{11}\varphi_1^2(\|x\|) \le v_{11}(t, x, w) \le \overline{\alpha}_{11}\varphi_2^2(\|x\|);$$
$$\underline{\alpha}_{22}\psi_1^2(\|x\|) \le v_{22}(t, x, w) \le \overline{\alpha}_{22}\psi_2^2(\|x\|);$$
$$\underline{\alpha}_{12}\varphi_1(\|x\|)\psi_1(\|x\|) \le v_{12}(t, x, w) \le \overline{\alpha}_{12}\varphi_2(\|x\|)\psi_2(\|x\|)$$

for all $x \in H$, $w \in Q$ and $t \ge 0$.

Lemma 5.1 *If all conditions of Assumption 5.1 are fulfilled and the matrices*

$$A_1 = \begin{pmatrix} \underline{\alpha}_{11} & \underline{\alpha}_{12} \\ \underline{\alpha}_{21} & \underline{\alpha}_{22} \end{pmatrix}, \quad \underline{\alpha}_{12} = \underline{\alpha}_{21},$$

$$A_2 = \begin{pmatrix} \overline{\alpha}_{11} & \overline{\alpha}_{12} \\ \overline{\alpha}_{21} & \overline{\alpha}_{22} \end{pmatrix}, \quad \overline{\alpha}_{12} = \overline{\alpha}_{21},$$

are positive definite, then the function

$$v(t,x,w) = \eta^{\mathrm{T}} U(t,x,w)\eta, \tag{15}$$

where $\eta = (\eta_1, \eta_2)^{\mathrm{T}}$, $\eta_i > 0$, *is positive definite and decreasing.*

Proof We introduce the notations

$$r = (\varphi_1(\|x\|), \psi_1(\|w\|))^{\mathrm{T}}, \quad q = (\varphi_2(\|x\|), \psi_2(\|w\|))^{\mathrm{T}}, \quad B = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}.$$

Under the conditions of Assumption 5.1 for the function (15) the bilateral estimation

$$r^{\mathrm{T}} B^{\mathrm{T}} A_1 B r \le \eta^{\mathrm{T}} U(t,x,w)\eta \le q^{\mathrm{T}} B^{\mathrm{T}} A_2 B q \tag{16}$$

holds.

By virtue of conditions of Lemma 5.1 it follows from the estimation (16) that the function $v(t,x,w)$ is positive definite and decreasing.

Assumption 5.2 There exist:

(1) functions $v_{11}(t,x)$, $v_{22}(t,w)$ and functions $v_{12}(t,x,w) = v_{21}(t,x,w)$;

(2) constants β_{ik}, $i = 1,2$, $k = 1,\dots,8$, and functions $\xi_1 = \xi_1(\|x\|)$ and $\xi_2 = \xi_2(\|w\|)$ of the K-class such that

(a) $D_t^+ v_{11}(t,x) + D_x^+ v_{11}(t,x)|_X \le \beta_{11}\xi_1^2$;

(b) $D_x^+ v_{11}(t,x)|_{g_1} \le \beta_{12}\xi_1^2 + \beta_{13}\xi_1\xi_3$;

(c) $D_t^+ v_{22}(t,w) + D_w^+ v_{22}(t,w)|_L \le \beta_{21}\xi_2^2$;

(d) $D_w^+ v_{22}(t,w)|_{g_2} \le \beta_{22}\xi_2^2 + \beta_{23}\xi_1\xi_2$;

(e) $D_t^+ v_{12(t,x,w)} + D_x^+ v_{12}(t,x,w)|_X \le \beta_{14}\xi_1^2 + \beta_{15}\xi_1\xi_2$;

(f) $D_w^+ v_{12}(t,x,w)|_L \le \beta_{24}\xi_1^2 + \beta_{25}\xi_1\xi_2$;

(g) $D_x^+ v_{12}(t,x,w)|_{g_1} \le \beta_{16}\xi_1^2 + \beta_{17}\xi_1\xi_2 + \beta_{18}\xi_2^2$;

(h) $D_w^+ v_{12}(t,x,w)|_{g_2} \le \beta_{26}\xi_1^2 + \beta_{27}\xi_1\xi_2 + \beta_{28}\xi_2^2$.

Lemma 5.2 *If all conditions of Assumption 5.2 are fulfilled and the matrix*

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \qquad c_{12} = c_{21},$$

with the elements

$$\begin{aligned} c_{11} &= \eta_1^2(\beta_{11} + \beta_{12}) + 2\eta_1\eta_2(\beta_{14} + \beta_{16} + \beta_{26}), \\ c_{22} &= \eta_2^2(\beta_{21} + \beta_{22}) + 2\eta_1\eta_2(\beta_{18} + \beta_{24} + \beta_{28}), \\ c_{12} &= \frac{1}{2}\,(\eta_1^2\beta_{13} + \eta_2^2\beta_{23}) + \eta_1\eta_2(\beta_{15} + \beta_{25} + \beta_{17} + \beta_{27}) \end{aligned}$$

is negative definite, then the derivative

$$D^+ v(t,x,w) = \eta^{\mathrm{T}} D^+ U(t,x,w)\eta$$

of the function $v(t,x,w)$ *is a negative definite function by virtue of the system (11), (12).*

Proof By virtue of the estimations (a) – (d) of Assumption 5.2 the estimation

$$D^+ v(t,x,w) \le p^{\mathrm{T}} C p$$

holds, where $p = (\xi_1(\|x\|), \xi_2(\|x\|))^{\mathrm{T}}$.

A definite negativeness of the derivative follows from the condition of Lemma 5.2.

Theorem 5.1 *If the two-component system (11), (12) is such that all conditions of Lemmas 5.1 and 5.2 are fulfilled, then the state of equilibrium* $x = 0$, $w = 0$ *of the system is uniform asymptotically stable.*

If in Assumption 5.1 $N_x = R^k$, $N_w = Q$, *functions* φ_i, ψ_i, ξ_i *belong to the KR-class and conditions of Lemmas 5.1, 5.2 are fulfilled, then the state of equilibrium* $x = 0$, $w = 0$ *of the system (11), (12) is uniform asymptotically stable in the whole.*

Proof Under the enumerated conditions the function $v(t, x, w)$ and its full derivative satisfy all conditions of Theorem 4.1. It proves the statement of Theorem 5.1.

Remark 5.1 If in estimations (a) – (d) of Assumption 5.2 we change the sign of the inequality for the opposite one and leave in the inequalities of Assumption 5.1 only estimation from below, then it isn't difficult to define conditions of instability of the state $x = 0$, $w = 0$ of the system (11), (12).

6 Concluding Remarks

Similar to Theorem 3.1 in the paper [7] the theorem was proved for discontinuous dynamical system. The mappings preserving stability in metric space were first considered by Thomas [10] and Hahn [3]. In the papers [8] and the book [9] and other mappings of the type were studied in the stability analysis of large-scale systems.

The application of multicomponent mapping $U(t, p)$: $R_+ \times X_1 \to X_2$ adds more flexibility to the approach to stability analysis of dynamical system in metric space, because this mapping admits a wider class of components for its elements $v_{ij}(t, p)$,.

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