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A Duality Principle in the Theory of Dynamical Systems

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Abstract: The aim of this paper is to formulate and illustrate a duality principle for dynamical systems. There is a one-to-one correspondence between causal (nonanticipative) systems, and the anticipative ones. Several cases are dealt with, based on the nature of the functional equations describing the dynamics.

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1 Introduction

The dynamical systems we shall consider in this paper will be described by functional equations of various types.

The duality principle we are going to formulate and illustrate establishes a one-to-one correspondence between the class of *causal systems*, and the class of *anticipative systems*. The first class is also known as abstract Volterra systems, while the second class contains the so-called anti-Volterra systems.

The principle of duality states that: *to any causal system, one can associate an anticipative systems, and vice-versa.*

Moreover, the mathematical treatment is basically the same for causal/ anticipative couples which are in correspondence.

The idea of formulating this duality principle came from writing our joint paper [3], in which the mathematical apparatus used in dealing with anticipative systems (the corresponding describing equations are with advanced argument), has revealed a striking

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resemblance with the one used when investigating causal systems (usually described by functional equations of Volterra type).

As we can expect, the initial state in the causal system becomes the terminal state in the anticipative systems, and vice-versa.

Apparently, the anticipative systems, which sometimes (see, for instance, Dubois [4, 5]) are called *anticipatory* systems, present interest in various applied areas, including some economic problems. Several proceedings volumes have been published, under the editorship of Dubois [4, 5]. They illustrate the significance of various types of anticipative systems, both theoretically and from the point of view of applications.

2 A Class of Discrete Systems

Let us consider a dynamical system with a finite number of states, say $x(t_i),\ i = 1, 2, \dots, n,$ with t_i an increasing sequence of reals. We assume $x(t_i) \in R$, $i = 1, 2, \dots, n$, even though we could deal with more general spaces than R, e.g., a Banach space E. We shall denote, for brievety, $x(t_i) = x_i,\ i = 1, 2, \dots, n$.

Let us further assume that the dynamics of the system is described by n equations of the form

$$\begin{aligned} &x_1 = f_1(x_1), \\ &x_2 = f_2(x_1, x_2), \\ &\dots\dots\dots\dots\dots\dots \\ &x_n = f_n(x_1, x_2, \dots, x_n). \end{aligned} \tag{1}$$

The particular form of the system (1) expresses the fact that we deal with a *causal system*. As each equation shows, the state of the system at the moment $t_k,\ 1 \le k \le n,$ depends only on the states at moments preceding or equal to t_k.

Now let us operate a change of variables $t_k = -\tau_{n-k+1},\ x_k = y_{n-k+1},\ k = 1, 2, \dots, n.$ Then, the system (1) becomes

$$\begin{aligned} &y_1 = f_n(y_n, y_{n-1}, \dots, y_1), \\ &y_2 = f_{n-1}(y_n, y_{n-1}, \dots, y_2), \\ &\dots\dots\dots\dots\dots\dots\dots \\ &y_n = f_1(y_n). \end{aligned} \tag{2}$$

From (2), we see that the system is of *anticipative* type (or, as sometimes called, *anticipatory*).

Since the times $t_k,\ 1 \le k \le n,$ form an increasing sequence, there follows that the new times $\tau_k,\ 1 \le k \le n,$ also form an increasing sequence: $\tau_1 < \tau_2 < \cdots < \tau_n$.

It is obvious that the systems (1) and (2) are identical, which tells us that from mathematical point of view, we have to solve the same problem for either of the associated causal and anticipative systems.

There are several questions rising from the above discussion related to the system (1), (2). Namely, the equations describing the dynamics of the system have been chosen in such a way that we deal with a “determined” system. In other words, we assume that the system (1) has a unique solution. This situation can be easily achieved. To take just an elementary example, we will assume that $|\partial f_k/\partial x_k| \le m_k < 1,\ k = 1, 2, \dots, n.$ This implies the existence of a unique real solution to the first equation; then substituting in the second equation this value for x_1, we will again determine a unique value for x_2, and so on. At each step we have to apply the Banach contraction mapping principle, in order to determine the unique value of the variable characterizing the state of the system. Many other conditions can be imposed in order to achieve the existence result specified above.

In case the system (1) has several solutions, say $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ and $(\bar{\bar{x}}_1, \bar{\bar{x}}_2, \dots, \bar{\bar{x}}_n)$, then we can define two anticipative systems the same way we have proceeded in the case of a unique solution. In other words, if there exist two (or several) causal systems described by the equations (1), we can accordingly associate two (or several) anticipative systems with "reverse" dynamics.

Another aspect to be considered corresponds to the situation when the system (1) is not determined, in the sense that some of the variables can be chosen arbitrarily (for instance, the n equations are not independent). What is, in such a case, the adequate manner to attach to (1) an anticipative system? Apparently, this is possible because if we assign values to some of the x_k's, the remaining equations still describe a causal system.

Finally, we would like to formulate an open problem (apparently) related to the topics discussed above. Namely, if the system (1) is replaced by a more general system of equations like $f_k(x_1, x_2, \dots, x_n) = 0$, $k = 1, 2, \dots, n$, *under what conditions can we state that they describe the dynamics of a causal system*?

3 Systems Described by Integral Equations

In this section we shall illustrate the duality principle in the case the dynamics of the system is described by an integral equation of anti-Volterra type. Therefore, we shall start with an anticipative system, and construct the causal system whose dynamics is determined by the same data as those of the given anticipative system.

The dynamical system under consideration in this section is defined by means of a function $x = x(t)$, $0 \le t \le T$, the values of x being taken in a Banach space E. The describing equation for the dynamics is of the form

$$x(t) = f(t) + \int_t^T k(t, s, x(s))\, ds, \tag{3}$$

with $f \in C([0, T], E)$, and $k(t, s, x)$ defined and continuous on $\Delta \times E$, with values in E, where $\Delta = \{(t, s) \colon\ 0 \le t \le s \le T\} \subset R^2$. If we also admit for $k(t, s, x)$ a Lipschitz type condition in $\Delta \times E$,

$$\|k(t, s, x) - k(t, s, y)\| \le L\|x - y\|, \quad L > 0, \tag{4}$$

then we get existence and uniqueness of the solution $x = x(t)$, $0 \le t \le T$, which is in $C([0, T], E)$.

The proof of existence and uniqueness of the solution to (3) can be conducted on the classical pattern, by the method of successive approximations

$$x_{n+1}(t) = f(t) + \int_t^T k(t, s, x_n(s))\, ds, \quad n \ge 0, \tag{5}$$

with $x_0(t) = f(t),\ t \in [0, T]$. It has been carried out in our paper [3].

The case when we have existence on the whole interval $[0, T]$, but not necessary uniqueness, has been also discussed in [3].

Now let $x = x(t)$ be a solution of (3) defined on $[0, T]$, in either case of uniqueness or nonuniqueness. In order to define the causal system corresponding to the anticipative system described by (3), we shall proceed as follows: we operate the change of variables $t = -\tau,\ s = -u$, and denote $f(-\tau) = \tilde{f}(\tau),\ x(-\tau) = y(\tau),\ \tau \in [-T, 0]$. Then (3) becomes

$$y(\tau) = \tilde{f}(\tau) + \int_{-T}^{\tau} k(-\tau, -u, y(u))\, du, \tag{6}$$

with $\tilde{f} \in C([-T, 0], E)$, and k defined on $\Delta \times E$, with values in E, where $\overline{\Delta} = \{(\tau, u)\colon -T \le u \le \tau \le 0\} \subset R^2$. Obviously, $k(-\tau, -u, y)$ satisfies a Lipschitz condition derived from (4).

It is obvious from (6) that the dynamics described by this equation is of causal type. It is possible to "shift" the considerations from the interval $[-T, 0]$, to any interval $[a, b] \subset R$.

We have again illustrated the duality principle, this time for continuous time dynamical systems for which the law of the dynamics is given by means of an integral equation of anti-Volterra type.

We shall see below that other types of dynamical systems can be reduced, in principle, to the case examined in this section.

4 A Case with General Causal Operators

In this section we will consider a Cauchy type problem, for a differential equation involving a linear causal operator, as well as a nonlinear part. More precisely, we shall deal with the equation

$$\dot{x}(t) = (Lx)(t) + (fx)(t), \quad t \in [0, T], \tag{7}$$

with the initial condition

$$x(0) = x^0 \in R^n, \quad n \ge 1. \tag{8}$$

The linear causal operator L is acting continuously on the space $L^2([0, T], R^n)$, while $f\colon L^2([0, T], R^n) \to L^2([0, T], R^n)$ is a continuous causal operator, generally nonlinear. It is understood that any solution we consider is of Carathéodory type, i.e., is in $AC([0, T], R^n)$ and satisfies the differential equation (7) a.e. on $[0, T]$.

For general properties of such equations we send the reader to the book [2] by C. Corduneanu. The formula of variation of parameters is given in the paper [6] by Yizeng Li.

As shown in the above mentioned references, the problem (7), (8) is equivalent to the integral equation of Volterra type

$$x(t) = X(t, 0)x^\circ + \int_0^t X(t, s)(fx)(s)\, ds, \tag{9}$$

where $X(t, s),\ 0 \le s \le t \le T$, is the Cauchy operator attached to the linear operator L in (7). In [2, 3], it is dealt with existence, and some properties are emphasized. See also the paper [7] by Mahdavi, in which L^2 is substituted by any L^p, $1 < p < \infty$.

The integral equation (9) is not exactly of the classical Volterra type, due to the presence of the operator f under the integral. In order to place ourselves in the classical framework, we shall assume that the operator f is a Niemytskii operator, i.e.,

$$(fx)(t) = F(t, x(t)), \quad t \in [0, T]. \tag{10}$$

In order to assure the fact that F is acting on $L^2([0,T];\, R^n)$, we can impose the growth condition

$$\|F(t,x)\| \le c\|x\| + a(t), \tag{11}$$

with $c > 0$ and $a \in L^2([0,T];\, R)$. Of course, we need some measurability conditions on F, and the Carathéodory assumptions are just adequate (i.e., continuity in x for almost all $t \in [0,T]$, and measurability in t for all $x \in R^n$).

With the choice (10) for the operator f in the equation (7), the integral equation (9) becomes

$$x(t) = X(t,0)x^\circ + \int_0^t X(t,s)F(s,x(s))\,ds. \tag{12}$$

Equation (12) is of classical Volterra type, and we can compare it with the equation (6). Due to the properties of $X(t,s)$, any solution of (12) belongs to the space $AC([0,T],\, R^n)$ of absolutely continuous maps, and satisfies the differential equation (7) almost everywhere on $[0,T]$.

There remains to write the integral equation of anticipative type, which describes the dynamics of the dual system associated to the system described by the equation (9), with $f = F$.

By the same substitution used in the preceding section, namely $t = -\tau$, $s = -u$, $x(-\tau) = y(\tau)$, the equation (12) leads to the integral equation of anticipative type on $[-T, 0]$,

$$y(\tau) = X(-\tau,0)x^\circ + \int_\tau^0 X(-\tau,-u)F(-u,\, y(u))\,du. \tag{13}$$

Conditions for existence/uniqueness of solution to the equation (13) can be found in the above mentioned references [2, 3].

Our aim was to illustrate once again the validity of the duality principle stated in this paper. In this case, the equations (12) and (13) describe the dynamics of the associated systems (causal and anticipative). At the same time, we have presented an example which relies on the use of general causal operators.

5 Conclusions and Open Problems

The examples discussed above show that whole classes of dynamical system can be used to illustrate the *duality principle* enunciated in this paper. What is really interesting, from mathematical point of view, is the fact that the mathematical apparatus is, basically, the same for the couple of associated systems. Moreover, from any result concerning the causal systems, one can derive a similar result for the associated anticipative systems.

We propose to the reader the following exercise: start with a result on causal systems in the book [2], and describe the corresponding result for the associated anticipative system. A first step would be to find the functional equation describing the dynamics of the anticipative system.

There seems to be a problem in dealing with the duality principle when we have infinite time in our initial system (causal or anticipative). In other words, when the time interval $[0, T]$ is replaced by the semi-axis $R_+ = [0, \infty)$. If for the describing functional equation we look for the so-called *transient* solutions, tending at infinity towards a stationary state, then the duality principle appears to be easy to be formulated. Another venue should be found when, for instance, we deal with an oscillatory solution of the describing functional equation. We mention this situation as an *open* problem.

Another open problem is to check the validity of the duality principle in case of dynamical systems whose dynamics is described by functional equations of the form $x(t) = f(t, x_t)$, with the usual notation $x_t(s) = x(t+s)$, $s \in [-T, 0]$. The case of differential equations with delay

$$\dot{x}(t) = f(t, x_t), \quad x(s) = x_0(s), \quad s \in [-T, 0],$$

is covered by the above mentioned functional equation $x(t) = f(t, x_t)$, with $x_0(s)$ assigned on $[-T, 0]$. One may succeed in this respect, by considering "dual" equations of the form $y(t) = \tilde{f}(t, y^t)$, with obvious meaning for y^t, namely $y^t(s) = y(t+s)$, $s \in [0, T]$.

A comprehensive approach, in order to produce an adequate framework for the statement and illustration of the duality principle, will require a more general concept of dynamical systems, in which the terms *causal* and *anticipative* make sense.

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