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On the Minimum Free Energy for the Ionosphere

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Abstract: Within the linear theory of the electromagnetism for the ionosphere we give a general closed expression for the minimum free energy in terms of Fourier-transformed quantities, when the integrated history of the electric field is chosen to characterize the state of the material. Another equivalent expression is derived and also used to study the particular case of a discrete spectrum model.

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1 Introduction

In a previous work [14] we have considered the problem of finding an explicit form for the minimum free energy of a linear conductor, characterized, in particular, by a constitutive equation for the current density expressed by a local functional of the history of the electric field.

This hereditary theory, which well describes the electromagnetic phenomena in the ionosphere [8, 9, 18], has been studied in particular in [10], where some thermodynamic potentials are derived, as well as the maximal free enthalpy and the maximal free energy; these representations depend on the choice of the state variables and several possibilities are considered. In [13] we have also considered a different constitutive equation, between the current density and the electric field, which associates the presence of memory effects with the actual action of the electric field; moreover, in particular, in this work we have derived the thermodynamic restrictions on the assumed constitutive equations.

As we have done in [14], still now we follow Golden's lines of [12], where the analogous problem is studied for a linear viscoelastic material in the scalar case, and the procedure used in [11] for the same problem always in viscoelasticity, see also [7, 15, 19].

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In [14], to characterize the state of the material we have chosen the instantaneous values of the electric and magnetic fields together with the history of the electric field. In the present work we assume the state made by the integrated history of the electric field in the place of its history, to give a different formulation of the problem, which requires, in particular, an appropriate new definition of the continuation of histories and processes.

In Section 2, we introduce some fundamental relationships and recall some useful results derived in previous works [13, 16]. Then, in Section 3, we give the definition of states and processes; moreover, we define the equivalence between integrated histories. Another definition of the equivalence of two integrated histories is also given by using the boundedness of the electromagnetic work, in Section 4 and Section 5. After, in Section 6, we derive the expression of the maximum recoverable work we can obtain by starting from a given state. Then, in Section 7, we derive a new form of the minimum free energy, which is applied in the last Section 8 to study the particular case of a discrete spectrum material.

2 Notation and Preliminaries

Let $\mathcal{B}$ be a rigid conducting material, whose electromagnetic behavior is characterized by these linear constitutive equations

$$\mathbf{D}(\mathbf{x},t) = \varepsilon \mathbf{E}(\mathbf{x},t), \qquad \mathbf{B}(\mathbf{x},t) = \mu \mathbf{H}(\mathbf{x},t), \tag{2.1}$$

$$\mathbf{J}(\mathbf{x},t) = \int_0^{+\infty} \alpha(s)\mathbf{E}^t(\mathbf{x},s)\, ds, \tag{2.2}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, $\mathbf{D}$ and $\mathbf{B}$ denote the electric displacement and the magnetic induction and $\mathbf{J}$ is the current density expressed in terms of the history of the electric field, $\mathbf{E}^t(\mathbf{x},s) = \mathbf{E}(\mathbf{x},\, t-s)$ $\forall\, s \in \mathbb{R}^+ = [0,+\infty)$, up to time t; moreover, the position vector is denoted by $\mathbf{x} \in \Omega$, the region occupied by the solid $\mathcal{B}$.

We suppose that the body is a homogeneous and isotropic material; hence, the dielectric constant $\varepsilon > 0,$ as well as the magnetic permeability $\mu > 0$ and the memory kernel α are constant in $\overline{\Omega}$. We assume that this relaxation function $\alpha\colon \mathbb{R}^+ \to R$ is such that $\alpha \in L^1(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$.

The region Ω occupied by the conductor $\mathcal{B}$ is a bounded and simply-connected domain of the three-dimensional Euclidean space, with a smooth boundary $\partial\Omega$ with the unit outward normal $\mathbf{n}$.

If we introduce the integrated history of $\mathbf{E}$, which is the function $\overline{\mathbf{E}}^t(\mathbf{x},\cdot)\colon\, \mathbb{R}^+ \to R^3$ defined by

$$\overline{\mathbf{E}}^t(\mathbf{x},s) = \int_0^s \mathbf{E}^t(\mathbf{x},\lambda)\, d\lambda = \int_{t-s}^t \mathbf{E}(\mathbf{x},\tau)\, d\tau, \tag{2.3}$$

the constitutive equation (2.2), with an integration by parts and taking into account the function α expressed by means of

$$\alpha(t) = \alpha_0 + \int_0^t \dot{\alpha}(\tau)\, d\tau \quad \forall\, t \in \mathbb{R}^+, \tag{2.4}$$

where α_0 is the initial value at time $t = 0$, with $\lim_{t\to+\infty} \alpha(t) = 0$, can be rewritten in the following equivalent form

$$\mathbf{J}(\mathbf{x}, t) = -\int_0^{+\infty} \alpha'(s)\bar{\mathbf{E}}^t(\mathbf{x}, s)\, ds. \tag{2.5}$$

Let us introduce the formal Fourier transform of any function $f\colon R \to R^n$ denoted by f_F and given by

$$f_F(\omega) = \int_{-\infty}^{+\infty} f(\xi)e^{-i\omega\xi}\, d\xi = f_+(\omega) + f_-(\omega), \tag{2.6}$$

where

$$f_+(\omega) = \int_0^{+\infty} f(\xi)e^{-i\omega\xi} d\xi, \qquad f_-(\omega) = \int_{-\infty}^{0} f(\xi)e^{-i\omega\xi} d\xi. \tag{2.7}$$

Besides the definitions of $f_\pm$, it is useful to consider the half-range Fourier cosine and sine transforms expressed by

$$f_c(\omega) = \int_0^{+\infty} f(\xi)\cos(\omega\xi)\, d\xi, \qquad f_s(\omega) = \int_0^{+\infty} f(\xi)\sin(\omega\xi)\, d\xi; \tag{2.8}$$

the definitions of f_c, f_s as well as f_+ hold also if the function f is defined on $\mathbb{R}^+$, while the definition of f_- stands for f defined on $\mathbb{R}^- = (-\infty, 0]$.

Furthermore, we observe that any function defined on $\mathbb{R}^+$ can be extended on R. If we identify functions on $\mathbb{R}^+$ with functions on R which vanish for any $s \in \mathbb{R}^{--}$, the strictly negative reals, we have

$$f_F(\omega) = f_c(\omega) - if_s(\omega). \tag{2.9}$$

The extension made with an even function, that is such that $f(\xi) = f(-\xi)\ \forall\, \xi < 0$, yields

$$f_F(\omega) = 2f_c(\omega), \tag{2.10}$$

while, when the function becomes an odd function, i.e. $f(\xi) = -f(-\xi)\ \forall\, \xi < 0$, such an extension gives

$$f_F(\omega) = -2if_s(\omega). \tag{2.11}$$

From the definitions (2.7) of $f_\pm$ it follows that we can consider these functions defined for any complex $z \in \mathbf{C}$, the complex plane; thus, $f_\pm(z)$ are analytic for $z \in \mathbf{C}^{(\mp)}$ where

$$\mathbf{C}^{(-)} = \left\{z \in \mathbf{C}\colon\ \operatorname{Im} z \in R^{--}\right\}, \quad \mathbf{C}^{(+)} = \left\{z \in \mathbf{C}\colon\ \operatorname{Im} z \in R^{++}\right\}. \tag{2.12}$$

This analyticity can be extended by hypothesis on $\mathbf{C}^{\mp}$ [12], where

$$\mathbf{C}^{-} = \left\{z \in \mathbf{C}\colon\ \operatorname{Im} z \in R^{-}\right\}, \quad \mathbf{C}^{+} = \left\{z \in \mathbf{C}\colon\ \operatorname{Im} z \in R_{+}\right\}, \tag{2.13}$$

thus including the real axis, which in (2.12) is excluded.

In the following we shall use the notation $f_{(\pm)}(z)$ to denote that the function has zeros and singularities in $\mathbf{C}^{\pm}$.

The thermodynamic restrictions on the constitutive equations were studied in [13, 16]; they yield the following constraint

$$\alpha_c(\omega) > 0 \quad \forall\,\omega \in R, \tag{2.14}$$

where we have added the hypothesis $\alpha_c(0) > 0$.

We recall some results of [10, 14]. Introducing the electric conductivity

$$\nu(t) = \int_0^t \alpha(\xi)\, d\xi, \tag{2.15}$$

it follows that its asymptotic value is

$$\nu_\infty = \int_0^{+\infty} \alpha(\xi)\, d\xi = \alpha_c(0) > 0. \tag{2.16}$$

Moreover, from the inverse Fourier transform

$$\alpha(t) = \frac{2}{\pi} \int_0^{+\infty} \alpha_c(\omega) \cos(\omega t)\, d\omega \tag{2.17}$$

we get

$$\alpha(0) = \frac{2}{\pi} \int_0^{+\infty} \alpha_c(\omega)\, d\omega > 0, \qquad \alpha'(0) \le 0. \tag{2.18}$$

Finally, we have

$$\alpha_s'(\omega) = -\omega\alpha_c(\omega), \quad \lim_{\omega\to\infty} \omega\alpha_s'(\omega) = -\lim_{\omega\to\infty} \omega^2\alpha_c(\omega) = \alpha'(0) \le 0 \tag{2.19}$$

if $\alpha'' \in L^2(\mathbb{R}^+)$ and $|\alpha'(0)| < +\infty$.

We assume

$$\alpha'(0) < 0. \tag{2.20}$$

Given a history $\mathbf{E}^t(s) = \mathbf{E}(t-s)$ for any $s \in R^+$, we consider its static continuation of duration $\tau \in R^{++}$ defined as

$$\mathbf{E}^{t_{(\tau)}} = \begin{cases} \mathbf{E}(t), & s \in [0,\tau], \\ \mathbf{E}^t(s-\tau), & s > \tau, \end{cases} \tag{2.21}$$

to which corresponds the following integrated history

$$\overline{\mathbf{E}}^{t+\tau}(s) = \begin{cases} \int_0^s \mathbf{E}(t)\, d\eta = s\mathbf{E}(t), & 0 \le s \le \tau, \\ \tau\mathbf{E}(t) + \int_0^{s-\tau} \mathbf{E}^t(\rho)\, d\rho, & s > \tau. \end{cases} \tag{2.22}$$

This static continuation yields a current density, which, taking into account (2.5) where the integrated history to be considered is expressed by (2.22), assumes the following form

$$\mathbf{J}(t+\tau) = \nu(\tau)\mathbf{E}(t) - \int_0^{+\infty} \alpha'(\tau+\rho)\overline{\mathbf{E}}^t(\rho)\, d\rho. \tag{2.23}$$

3 States, Processes and Equivalent Integrated Histories

The constitutive equations (2.1) – (2.2), assumed to describe the electromagnetic behavior of our conductor $\mathcal{B}$, characterize a simple material [3, 4], whose states and processes can be defined as follows.

We observe that all the functions introduced in the previous section depend on $\mathbf{x} \in \Omega$ and $t \in \mathbb{R}^+$; in the following we shall understand the dependence on $\mathbf{x}$, which, therefore will not be written.

Contrary to the choice made in [14], we now consider the electromagnetic state of $\mathcal{B}$ given by

$$\sigma(t) = (\mathbf{E}(t),\, \mathbf{H}(t),\, \overline{\mathbf{E}}^t), \tag{3.1}$$

where the integrated history of the electric field is chosen to give the memory effects on the instantaneous values of the current density. We denote by Σ the set of the admissible states of $\mathcal{B}$.

The electromagnetic process is defined by the function $P\colon\, [0,d) \to R^3 \times R^3$, supposed piecewise continuous on the time interval $[0,d) \subset \mathbb{R}^+$ and expressed by

$$P(\tau) = (\dot{\mathbf{E}}_P(\tau),\, \dot{\mathbf{H}}_P(\tau)) \quad \forall\, \tau \in [0,d), \tag{3.2}$$

where $d \in R_+$ is said the duration of the process and $\dot{\mathbf{E}}_P(\tau)$ and $\dot{\mathbf{H}}_P(\tau)$ are the time derivatives of the electric and magnetic fields $\mathbf{E}_P$ and $\mathbf{H}_P$ at any instant of the time interval. The set of all admissible processes is denoted by Π. Sometimes, we shall consider the restriction of a process P to the interval $[t_1, t_2) \subset [0,d)$, denoted by $P_{[t_1,t_2)}$.

Given any initial state $\sigma^i \in \Sigma$ and a process $P \in \Pi$, the evolution function $\rho\colon\, \Sigma \times \Pi \to \Sigma$ provides the final state $\sigma^f = \rho(\sigma^i, P)$. If $\sigma(0)$ is the initial state and we apply $P_{[0,t)}$, the evolution function yields the state $\sigma(t) = \rho(\sigma(0), P_{[0,t)})\ \forall\, t \in [0,d]$; in particular, the pair (σ, P) is called a cycle if $\sigma(d) = \rho(\sigma(0),\, P) = \sigma(0)$.

The response of the material is the function

$$U(t) = (\mathbf{D}(t),\, \mathbf{B}(t),\, \mathbf{J}(t)). \tag{3.3}$$

In (3.3) the instantaneous values of $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{J}$ obviously depend on the pair (σ, P); therefore, the output function $U(t)$ is a function $\widetilde{U}\colon\, \Sigma \times \Pi \to R^3 \times R^3 \times R^3$ such that

$$U = \widetilde{U}(\sigma, P). \tag{3.4}$$

In (3.3) the triplet $(\mathbf{D}(t), \mathbf{B}(t), \mathbf{J}(t))$ is expressed in terms of (2.1) and (2.2), the last of which allows us to introduce the linear functional $\tilde{\mathbf{J}}\colon\, \Gamma \to R^3$ such that

$$\tilde{\mathbf{J}}(\overline{\mathbf{E}}^t) = -\int_0^{+\infty} \alpha'(s)\overline{\mathbf{E}}^t(s)\, ds, \tag{3.5}$$

which is defined in the following function space

$$\Gamma = \left\{ \overline{\mathbf{E}}^t\colon\ (0,+\infty) \to R^3;\ \ \left| \int\limits_0^{+\infty} \alpha'(s+\tau)\overline{\mathbf{E}}^t(s)\,ds \right| < +\infty \quad \forall\, \tau \geq 0 \right\}. \tag{3.6}$$

The process (3.2) is defined in the time interval $[0,d)$, but its application can be done at any initial instant $t \geq 0$; therefore, we distinguish the following two cases.

If the initial state is $\sigma(0) = (\mathbf{E}_*(0),\, \mathbf{H}_*(0),\, \overline{\mathbf{E}}_*^0) \in \Sigma$ and the process is applied at time $t = 0$, then the process is $P(t) = (\dot{\mathbf{E}}_P(t),\, \dot{\mathbf{H}}_P(t)) \in \Pi$, since in this case $\tau \equiv t$ in (3.2). Thus, the process yields a family of states $\sigma(t) = (\mathbf{E}(t),\, \mathbf{H}(t),\, \overline{\mathbf{E}}^t)\ \forall\, t \in [0,d]$, where

$$\mathbf{E}(t) = \mathbf{E}_*(0) + \int\limits_0^t \dot{\mathbf{E}}_P(\rho)\,d\rho, \qquad \mathbf{H}(t) = \mathbf{H}_*(0) + \int\limits_0^t \dot{\mathbf{H}}_P(\rho)\,d\rho, \tag{3.7}$$

that is $\mathbf{E}_P(\tau) \equiv \mathbf{E}_P(t) = \mathbf{E}(t)$ and similarly $\mathbf{H}_P(\tau) \equiv \mathbf{H}_P(t) = \mathbf{H}(t)$, and

$$\overline{\mathbf{E}}^t(\tau') = \begin{cases} \int\limits_{t-\tau'}^{t} \mathbf{E}(s)\,ds, & 0 \leq \tau' < t, \\ \overline{\mathbf{E}}_*^0(\tau'-t) + \int\limits_0^t \mathbf{E}(s)\,ds, & \tau' \geq t, \end{cases} \tag{3.8}$$

$\mathbf{E}(s)$ being given by $(3.7)_1$.

When a process P is applied at time $t > 0$ and the initial state is $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), \overline{\mathbf{E}}^t)$, we relate the process $P(\tau) = (\dot{\mathbf{E}}_P(\tau),\, \dot{\mathbf{H}}_P(\tau))\ \forall\, \tau \in [0,d)$, d being its duration, to

$$\mathbf{E}_P\colon \quad (0,d] \to R^3, \quad \mathbf{E}_P(\tau) = \mathbf{E}(t) + \int\limits_0^\tau \dot{\mathbf{E}}_P(s')\,ds', \tag{3.9}$$

$$\mathbf{H}_P\colon \quad (0,d] \to R^3, \quad \mathbf{H}_P(\tau) = \mathbf{H}(t) + \int\limits_0^\tau \dot{\mathbf{H}}_P(s')\,ds', \tag{3.10}$$

for every $\tau \in (0,d]$. Consequently, the process P induces a prolongation of the initial integrated history $\overline{\mathbf{E}}^t$, related to the final value $\mathbf{E}_P(d)$ and the continuation of the initial $\overline{\mathbf{E}}^t$ by means of $\mathbf{E}_P(\tau) = \mathbf{E}(t+\tau)$, with $t + \tau \leq t + d$, as follows

$$\overline{\mathbf{E}}^{t+d}(s) = (\mathbf{E}_P * \overline{\mathbf{E}})^{t+d}(s) = \begin{cases} \overline{\mathbf{E}}_P^d(s) = \int\limits_0^s \overline{\mathbb{E}}_P^d(\rho)\,d\rho, & 0 \leq s < d, \\ \overline{\mathbf{E}}_P^d(d) + \overline{\mathbf{E}}^t(s-d), & s \geq d, \end{cases} \tag{3.11}$$

where the integrated histories

$$\overline{\mathbf{E}}_P^d(d) = \int\limits_0^d \mathbf{E}_P(\eta)\,d\eta, \qquad \overline{\mathbf{E}}^t(s-d) = \int\limits_{t-(s-d)}^{t} \mathbf{E}(\xi)\,d\xi \tag{3.12}$$

are related to the process in $[0, d)$ and the initial integrated history, respectively.

We are able to evaluate the final value of the current density assumed by means of a process P applied to any initial state $\sigma(t)$ during the time interval $[0, d)$, i.e. $\tilde{\mathbf{J}}((\mathbf{E}_P * \overline{\mathbf{E}})^{t+d})$, taking account of (3.5) and (3.11).

If the restriction $P_{[0,\tau)}$ is applied to the initial state $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), \overline{\mathbf{E}}^t)$, from (3.5), considering the integral between zero and infinity as the sum of two integrals, the first of which between zero and τ and the second one between τ and infinity, and substituting d with τ in (3.11), we have

$$\begin{aligned}
\tilde{\mathbf{J}}(\overline{\mathbf{E}}^{t+\tau}) &= -\int_0^{+\infty} \alpha'(s)(\mathbf{E}_P * \overline{\mathbf{E}})(t+\tau-s)\, ds \\
&= -\int_0^{\tau} \alpha'(\eta)\overline{\mathbf{E}}_P^{\tau}(\eta)\, d\eta - \int_{\tau}^{+\infty} \alpha'(s)[\overline{\mathbf{E}}_P^{\tau}(\tau) + \overline{\mathbf{E}}^t(s-\tau)]ds \qquad (3.13) \\
&= \alpha(\tau)\overline{\mathbf{E}}_P^{\tau}(\tau) - \int_0^{\tau} \alpha'(\eta)\overline{\mathbf{E}}_P^{\tau}(\eta)\, d\eta - \int_{\tau}^{+\infty} \alpha'(s)\overline{\mathbf{E}}^t(s-\tau)\, ds.
\end{aligned}$$

Definition 3.1 Two states $\sigma_i \in \Sigma$, $i = 1, 2$, are said to be equivalent if

$$\widetilde{U}(\sigma_1, P) = \widetilde{U}_2(\sigma, P) \quad \forall\, P \in \Pi. \qquad (3.14)$$

In other words, two states are equivalent if the response of the material is the same whatever may be the applied admissible process. Therefore, it follows that two initial states which produce the same response during the application of any process are indistinguishable.

This definition [15] yields an equivalent relation $\mathcal{R}$ in the space Σ of the states, which induces the introduction of the quotient space $\Sigma_{\mathcal{R}}$, whose elements are the classes of equivalent states denoted by $\sigma_{\mathcal{R}}$.

Definition 3.2 A state of the body is said minimal if it is characterized by the minimum set of data.

We only observe that Definition 3.1 implies the equality of some quantities present in the definition of the states; these conditions are absorbed by the following definition of equivalence between two integrated histories of the electric field.

Definition 3.3 Given two states $(\mathbf{E}_i(t), \mathbf{H}_i(t), \overline{\mathbf{E}}_i^t)$, $i = 1, 2$, corresponding to the same value of the magnetic field, $\mathbf{H}_i(t) = \mathbf{H}(t)$, $i = 1, 2$, the integrated histories of the electric field $\overline{\mathbf{E}}_i^t$ $(i = 1, 2)$ are called equivalent if

$$\mathbf{E}_1(t) = \mathbf{E}_2(t), \quad \tilde{\mathbf{J}}((\mathbf{E}_P * \overline{\mathbf{E}}_1)^{t+\tau}) = \tilde{\mathbf{J}}((\mathbf{E}_P * \overline{\mathbf{E}}_2)^{t+\tau}) \qquad (3.15)$$

for every $\mathbf{E}_P\colon (0, \tau] \to R^3$ and for every $\tau > 0$, whatever may be $\mathbf{H}_P\colon (0, \tau] \to R^3$.

This definition characterizes the integrated histories associated to the same current density; moreover, we note that it has no effects on the magnetic field $\mathbf{H}_P(\tau)$, whose values are independent of $\overline{\mathbf{E}}_i^t$, $i = 1, 2$.

The zero integrated history is the particular history such that $\overline{\mathbf{E}}^t(s) = \bar{\mathbf{0}}^\dagger(s) = \mathbf{0}$ $\forall\, s \in \mathbb{R}^+$; its continuation by means of the process $P_{[0,\tau)}$, applied to the initial state $\sigma(t) = (\mathbf{E}(t),\, \mathbf{H}(t),\, \bar{\mathbf{0}}^\dagger)$ is given by

$$(\mathbf{E}_P * \bar{\mathbf{0}}^\dagger)^{t+\tau}(s) = \begin{cases} \overline{\mathbf{E}}_P^\tau(s), & 0 \le s < \tau, \\ \overline{\mathbf{E}}_P^\tau(\tau), & s \ge \tau. \end{cases} \tag{3.16}$$

Taking account of (3.13), it follows that an integrated history $\overline{\mathbf{E}}^t$ is equivalent to the zero integrated history $\bar{\mathbf{0}}^\dagger$ if

$$\int_\tau^{+\infty} \alpha'(s)\overline{\mathbf{E}}^t(s-\tau)\, ds = \int_0^{+\infty} \alpha'(\tau+\rho)\overline{\mathbf{E}}^t(\rho)\, d\rho = \mathbf{0}. \tag{3.17}$$

This condition yields a new equivalence relation, which characterizes the same equivalence between two integrated histories of Definition 3.3. In fact, two integrated histories $\overline{\mathbf{E}}_i^t$ $(i = 1, 2)$, equivalent in the sense of Definition 3.3, must satisfy (3.15), from which it follows that this equality

$$\int_\tau^{+\infty} \alpha'(s)\overline{\mathbf{E}}_1^t(s-\tau)\, ds = \int_\tau^{+\infty} \alpha'(s)\overline{\mathbf{E}}_2^t(s-\tau)\, ds \tag{3.18}$$

must hold. Introducing the difference $\overline{\mathbf{E}}^t(s-\tau) = \overline{\mathbf{E}}_1^t(s-\tau) - \overline{\mathbf{E}}_2^t(s-\tau)$, we see that (3.18) coincides with (3.17), which assures the equivalence between the integrated history given by this difference and the zero integrated history.

4 Electromagnetic Work

The electromagnetic work done on a process $P(\tau) = (\dot{\mathbf{E}}_P(\tau),\, \dot{\mathbf{H}}_P(\tau))$, defined for every $\tau \in [0, d)$, starting from the initial state $\sigma^i(t) = (\mathbf{E}_i(t),\, \mathbf{H}_i(t),\, \overline{\mathbf{E}}_i^t)$, is expressed by [1, 2, 16]

$$\begin{aligned} \widetilde{W}(\mathbf{E}_i(t), \mathbf{H}_i(t), \overline{\mathbf{E}}_i^t; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) &= \int_t^{t+d} \Big[\dot{\mathbf{D}}(\mathbf{E}(\xi)) \cdot \mathbf{E}(\xi) + \dot{\mathbf{B}}(\mathbf{H}(\xi)) \cdot \mathbf{H}(\xi) \\ &\qquad + \tilde{\mathbf{J}}((\mathbf{E}_P * \overline{\mathbf{E}}_i)^\xi) \cdot \mathbf{E}(\xi)\Big] d\xi \qquad (4.1) \\ &= \int_0^d \Big[\dot{\mathbf{D}}(\mathbf{E}_P(\tau)) \cdot \mathbf{E}_P(\tau) + \dot{\mathbf{B}}(\mathbf{H}_P(\tau)) \cdot \mathbf{H}_P(\tau) + \tilde{\mathbf{J}}((\mathbf{E}_P * \overline{\mathbf{E}}_i)^{t+\tau}) \cdot \mathbf{E}_P(\tau)\Big] d\tau, \end{aligned}$$

where the subscript P indicates that the fields are expressed by (3.9) – (3.10) in terms of $\tau \in (0, d]$.

In (4.1) we have considered $\widetilde{W}$ as a function of all the quantities which characterize the states and the processes of $\mathcal{B}$; therefore, the work is a function of the couple $(\sigma(t), P)$. Taking account of (2.1) and (2.5), we can give (4.1) the following form

$$W(\sigma(t), P) = \frac{1}{2}\left[\varepsilon\mathbf{E}^2(t+d) + \mu\mathbf{H}^2(t+d)\right] - \frac{1}{2}\left[\varepsilon\mathbf{E}^2(t) + \mu\mathbf{H}^2(t)\right] - \int_t^{t+d}\int_0^{+\infty} \alpha'(s)(\mathbf{E}_P * \overline{\mathbb{E}}_i)^{\xi}(s)\,ds \cdot \mathbf{E}(\xi)\,d\xi, \tag{4.2}$$

where $(\mathbf{E}_P * \overline{\mathbb{E}}_i)^{\xi}(s)$ is expressed by (3.11).

It is interesting to distinguish the part of the work due only to the process. For this purpose, let us examine the particular case, corresponding to the initial state $\sigma(0) = (\mathbf{0}, \mathbf{0}, \bar{\mathbf{0}}^{\dagger})$, i.e. when $\mathbf{E}(0) = \mathbf{0}$, $\mathbf{H}(0) = \mathbf{0}$, $\overline{\mathbf{E}}^{0}(s) = \bar{\mathbf{0}}^{\dagger}(s) = \mathbf{0}$. Thus, the process P of duration d is applied at time $t = 0$ and yields the ensuing fields (3.7)–(3.8), which now reduce to

$$\mathbf{E}_0(t) = \int_0^t \dot{\mathbf{E}}_P(s)\,ds, \qquad \mathbf{H}_0(t) = \int_0^t \dot{\mathbf{H}}_P(s)\,ds \tag{4.3}$$

and

$$(\mathbf{E}_0 * \bar{\mathbf{0}}^{\dagger})^t(s) = \begin{cases} \overline{\mathbf{E}}_0^t(s), & 0 \le s < t, \\ \overline{\mathbf{E}}_0^t(t), & s \ge t. \end{cases} \tag{4.4}$$

Definition 4.1 A process $P = (\dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P)$ of duration d, applied at time $t = 0$ and related to $\mathbf{E}_0(t)$, $\mathbf{H}_0(t)$ and $(\mathbf{E}_0 * \bar{\mathbf{0}}^{\dagger})^t$, given by (4.3)–(4.4), is said to be a finite work process if

$$\widetilde{W}(\mathbf{0}, \mathbf{0}, \mathbf{0}^{\dagger}; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) = \int_0^d \Big[\dot{\mathbf{D}}(\mathbf{E}_0(t)) \cdot \mathbf{E}_0(t) + \dot{\mathbf{B}}(\mathbf{H}_0(t)) \cdot \mathbf{H}_0(t) + \tilde{\mathbf{J}}((\mathbf{E}_0 * \bar{\mathbf{0}}^{\dagger})^t) \cdot \mathbf{E}_0(t)\Big]\,dt < +\infty. \tag{4.5}$$

Lemma 4.1 *The work considered in Definition 4.1 satisfies the inequality*

$$\widetilde{W}(\mathbf{0}, \mathbf{0}, \mathbf{0}^{\dagger};\, \dot{\mathbf{E}}_P,\, \dot{\mathbf{H}}_P) > 0. \tag{4.6}$$

Proof The work (4.5), taking into account (4.3)–(4.4), becomes

$$\widetilde{W}(\mathbf{0}, \mathbf{0}, \mathbf{0}^{\dagger}; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) = \frac{1}{2}[\varepsilon\mathbf{E}_0^2(d) + \mu\mathbf{H}_0^2(d)] - \int_0^d \left[\int_0^t \alpha'(s)\overline{\mathbf{E}}_0^t(s)\,ds + \int_t^{+\infty} \alpha'(s)\overline{\mathbf{E}}_0^t(t)\,ds\right] \cdot \mathbf{E}_0(t)\,dt, \tag{4.7}$$

which, integrating by parts in the first integral and evaluating the second one, reduces to

$$\widetilde{W}(\mathbf{0}, \mathbf{0}, \mathbf{0}^{\dagger}; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) = \frac{1}{2}[\varepsilon\mathbf{E}_0^2(d) + \mu\mathbf{H}_0^2(d)] + \int_0^d\int_0^t \alpha(s)\mathbf{E}_0^t(s)\,ds \cdot \mathbf{E}_0(t)\,dt. \tag{4.8}$$

Application of the Plancherel theorem, on assuming that for any $t > d$ the functions in (4.8) are equal to zero, allows us to transform the integral in (4.8) as follows

$$\int_0^d \int_0^t \alpha(s)\mathbf{E}_0^t(s)\,ds \cdot \mathbf{E}_0(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \alpha_F(\omega)\mathbf{E}_{0_F}(\omega)\cdot \mathbf{E}_{0_F}^*(\omega)\,d\omega, \tag{4.9}$$

where * denotes the complex conjugate. The functions we have transformed are defined on $\mathbb{R}^+$ and equal to zero on R^{--}; therefore we can use (2.9), which expresses the Fourier transforms in terms of the corresponding cosine and sine transforms, which are even and odd functions, respectively. Thus the integral in (4.9) reduces to

$$\int_{-\infty}^{+\infty} \alpha_F(\omega)\mathbf{E}_{0_F}(\omega)\cdot \mathbf{E}_{0_F}^*(\omega)\,d\omega = \int_{-\infty}^{+\infty} \alpha_c(\omega)[\mathbf{E}_{0_c}^2(\omega) + \mathbf{E}_{0_s}^2(\omega)]d\omega > 0, \tag{4.10}$$

by virtue of (2.14), and hence the positiveness of the work follows.

Let the duration of a process be $d < +\infty$, the process can be defined on $\mathbb{R}^+$ by assuming $P(\tau) = (\dot{\mathbf{E}}_P(\tau), \dot{\mathbf{H}}_P(\tau)) = (\mathbf{0},\mathbf{0})\ \forall\,\tau \ge d$; we assume $\mathbf{E}_P(\tau) = \mathbf{0}$, $\mathbf{H}_P(\tau) = \mathbf{0}$ $\forall\,\tau > d$. Then, taking into account that the initial integrated history is zero, (4.5) gives

$$\begin{aligned}
\widetilde{W}(\mathbf{0},\mathbf{0},\mathbf{0}^\dagger;\dot{\mathbf{E}}_P,\,\dot{\mathbf{H}}_P) &= \int_0^d [\varepsilon\dot{\mathbf{E}}_0(t)\cdot\mathbf{E}_0(t) + \mu\dot{\mathbf{H}}_0(t)\cdot\mathbf{H}_0(t)]\,dt \\
&\quad - \int_0^{+\infty}\left[\int_0^{\eta} \alpha'(s)\overline{\mathbf{E}}_P^{\eta}(s)\,ds + \int_{\eta}^{+\infty} \alpha'(s)\overline{\mathbf{E}}_P^{\eta}(\eta)\,ds\right]\cdot \mathbf{E}_P(\eta)\,d\eta \\
&= \frac{1}{2}[\varepsilon\mathbf{E}_0^2(d) + \mu\mathbf{H}_0^2(d)] + \int_0^{+\infty}\int_0^{\eta} \alpha(s)\mathbf{E}_P^{\eta}(s)\,ds\cdot\mathbf{E}_P(\eta)\,d\eta \\
&= \frac{1}{2}[\varepsilon\mathbf{E}_0^2(d) + \mu\mathbf{H}_0^2(d)] + \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} \alpha(|\eta-\rho|)\mathbf{E}_P(\rho)\cdot\mathbf{E}_P(\eta)\,d\rho d\eta.
\end{aligned} \tag{4.11}$$

Here the last integral can be transformed by applying the Plancherel theorem and taking into account that the Fourier transform of the even function $\alpha(|\eta-\rho|)$ can be written in terms of the Fourier cosine transform; thus, we get

$$\widetilde{W}(\mathbf{0},\mathbf{0},\mathbf{0}^\dagger;\,\dot{\mathbf{E}}_P,\,\dot{\mathbf{H}}_P) = \frac{1}{2}[\varepsilon\mathbf{E}_0^2(d) + \mu\mathbf{H}_0^2(d)] + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \alpha_c(\omega)\mathbf{E}_{P+}(\omega)\cdot\mathbf{E}_{P+}^*(\omega)\,d\omega. \tag{4.12}$$

With this result we can introduce the function space

$$\widetilde{H}_\alpha(\mathbb{R}^+, R^3) = \left\{\mathbf{g}\colon\ \mathbb{R}^+ \to R^3;\quad \int_{-\infty}^{+\infty} \alpha_c(\omega)\mathbf{g}_+(\omega)\cdot\mathbf{g}_+^*(\omega)\,d\omega < +\infty\right\},$$

which characterizes the finite work processes. We can consider as the space of the processes, to which $\mathbf{E}_P$ is related, the Hilbert space $H_\alpha(\mathbb{R}^+, R^3)$, obtained by the complation of $\widetilde{H}_\alpha$ with respect to the norm corresponding to the inner product

$$(\mathbf{g}_1, \mathbf{g}_2)_\alpha = \int_{-\infty}^{+\infty} \alpha_c(\omega)\mathbf{g}_{1+}(\omega) \cdot \mathbf{g}^*_{2+}(\omega)\, d\omega.$$

Let us now consider the general case, when the initial state is $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), \overline{\mathbf{E}}^t)$ and the work done on any process P, of duration $d < +\infty$, supposed to be zero for any $\tau \geq d$ and related to $\mathbf{E}_P(\tau) = \mathbf{0}$, $\mathbf{H}_P(\tau) = \mathbf{0}\ \forall\, \tau > d$, is given by (4.1), which yields

$$\begin{aligned} \widetilde{W}(\mathbf{E}(t), \mathbf{H}(t), \overline{\mathbf{E}}^t; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) &= \int_0^d [\varepsilon \dot{\mathbf{E}}_P(\tau) \cdot \mathbf{E}_P(\tau) + \mu \dot{\mathbf{H}}_P(\tau) \cdot \mathbf{H}_P(\tau)]\, d\tau \\ + \int_0^{+\infty} \left[\alpha(\tau)\overline{\mathbf{E}}^\tau_P(\tau) - \int_0^\tau \alpha'(s)\overline{\mathbf{E}}^\tau_P(s)\, ds - \int_0^{+\infty} \alpha'(\tau+\xi)\overline{\mathbf{E}}^t(\xi)\, d\xi \right] &\cdot \mathbf{E}_P(\tau)\, d\tau \end{aligned} \tag{4.13}$$

by using (3.13).

Evaluating the first integral and integrating by parts in the second one, (4.13) becomes

$$\begin{aligned} \widetilde{W}(\mathbf{E}(t), \mathbf{H}(t), \overline{\mathbf{E}}^t; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) &= \frac{1}{2}[\varepsilon \mathbf{E}^2_P(d) + \mu \mathbf{H}^2_P(d)] - \frac{1}{2}[\varepsilon \mathbf{E}^2_P(0) + \mu \mathbf{H}^2_P(0)] \\ &+ \int_0^{+\infty} \left[\int_0^\tau \alpha(\tau - \eta)\mathbf{E}_P(\eta)\, d\eta - \int_0^{+\infty} \alpha'(\tau+\xi)\overline{\mathbf{E}}^t(\xi)\, d\xi \right] \cdot \mathbf{E}_P(\tau)\, d\tau. \end{aligned} \tag{4.14}$$

Now, we put

$$\mathbf{I}(\tau, \overline{\mathbf{E}}^t) = \int_0^{+\infty} \alpha'(\tau+\xi)\overline{\mathbf{E}}^t(\xi)\, d\xi, \qquad \tau \geq 0. \tag{4.15}$$

and note that $\mathbf{I}(\tau, \overline{\mathbf{E}}^t)$ has the regularity induced by (2.23) where the continuation has the duration τ; moreover, we can evaluate its Fourier transform, which is given by

$$\mathbf{I}_+(\omega, \overline{\mathbf{E}}^t) = \int_0^{+\infty} e^{-i\omega\tau}\mathbf{I}(\tau, \overline{\mathbf{E}}^t)\, d\tau, \tag{4.16}$$

since $\mathbf{I}(\tau, \overline{\mathbf{E}}^t)$ is defined on $\mathbb{R}^+$.

Thus, (4.14) can be written as

$$\begin{aligned} \widetilde{W}(\mathbf{E}(t), \mathbf{H}(t), \overline{\mathbf{E}}^t; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P) &= \frac{1}{2}\left\{\varepsilon \mathbf{E}^2_P(d) + \mu \mathbf{H}^2_P(d) - [\varepsilon \mathbf{E}^2_P(0) + \mu \mathbf{H}^2_P(0)]\right\} \\ &+ \int_0^{+\infty} \left[\frac{1}{2} \int_0^{+\infty} \alpha(|\tau - \eta|)\mathbf{E}_P(\eta)\, d\eta - \mathbf{I}(\tau, \overline{\mathbf{E}}^t) \right] \cdot \mathbf{E}_P(\tau)\, d\tau, \end{aligned} \tag{4.17}$$

or using Plancherel's theorem, in the following equivalent form

$$\widetilde{W}(\mathbf{E}(t),\mathbf{H}(t),\overline{\mathbf{E}}^t;\dot{\mathbf{E}}_P,\dot{\mathbf{H}}_P) = \frac{1}{2}\left\{\varepsilon\mathbf{E}^2(t+d)+\mu\mathbf{H}^2(t+d)-[\varepsilon\mathbf{E}^2(t)+\mu\mathbf{H}^2(t)]\right\} + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\alpha_c(\omega)\mathbf{E}_{P+}(\omega)\cdot\mathbf{E}^*_{P+}(\omega)\,d\omega - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathbf{I}_+(\omega,\bar{\mathbf{E}}^t)\cdot\mathbf{E}^*_{P+}(\omega)\,d\omega. \tag{4.18}$$

5 Equivalence Between Two Integrated Histories Done by Means of the Work

In Section 3 we have called equivalent two integrated histories $\overline{\mathbf{E}}_1^t$ and $\overline{\mathbf{E}}_2^t$ which yield the same current density when they are subjected to the same process; hence, because of the equality imposed to the values of the electric and magnetic fields at the initial instant of the process, we have the same response of the material. An analogous equivalence relation between two integrated histories may be introduced in terms of the work.

Definition 5.1 Let $(\mathbf{E}_i,\mathbf{H}_i,\overline{\mathbf{E}}_i^t)$, $i=1,2$, be two given states of the electromagnetic material $\mathcal{B}$, the two integrated histories $\overline{\mathbf{E}}_i^t$ $(i=1,2)$ are called w-equivalent if the equality

$$\widetilde{W}(\mathbf{E}_1(t),\,\mathbf{H}_1(t),\,\overline{\mathbf{E}}_1^t;\,\dot{\mathbf{E}}_P,\dot{\mathbf{H}}_P) = \widetilde{W}(\mathbf{E}_2(t),\,\mathbf{H}_2(t),\,\overline{\mathbf{E}}_2^t;\,\dot{\mathbf{E}}_P,\dot{\mathbf{H}}_P) \tag{5.1}$$

holds for every $\dot{\mathbf{E}}_P\colon\,[0,\tau)\to R^3$, $\dot{\mathbf{H}}_P\colon\,[0,\tau)\to R^3$ and for every $\tau>0$.

The equivalence of the two definitions we have given follows easily from this theorem.

Theorem 5.1 *For any electromagnetic material described by the constitutive equations (2.1) and (3.5), two integrated histories of the electric field are equivalent in the sense of Definition 3.3 if and only if they are w-equivalent.*

Proof If two integrated histories $\overline{\mathbf{E}}_i^t$ $(i=1,2)$ are equivalent in the sense of Definition 3.3, then, for every $\mathbf{E}_P\colon\,(0,d]\to R^3$, $\mathbf{H}_P\colon\,(0,d]\to R^3$ and for every d, the works done on the process, of duration d and related to $\mathbf{E}_P$ and $\mathbf{H}_P$ by (3.9) – (3.10), applied to the states $(\mathbf{E}_1(t),\,\mathbf{H}_1(t),\,\overline{\mathbf{E}}_1^t)$ and $(\mathbf{E}_2(t),\,\mathbf{H}_2(t),\,\overline{\mathbf{E}}_2^t)$, characterized by $\mathbf{E}_1(t)=\mathbf{E}_2(t)$ and $\mathbf{H}_1(t)=\mathbf{H}_2(t)$, are the same since we have

$$\int_0^d[\dot{\mathbf{D}}(\mathbf{E}_P(\tau))\cdot\mathbf{E}_P(\tau)+\dot{\mathbf{B}}(\mathbf{H}_P(\tau))\cdot\mathbf{H}_P(\tau)+\tilde{\mathbf{J}}((\mathbf{E}_P*\overline{\mathbf{E}}_1)^{t+\tau})\cdot\mathbf{E}_P(\tau)]d\tau = \int_0^d[\dot{\mathbf{D}}(\mathbf{E}_P(\tau))\cdot\mathbf{E}_P(\tau)+\dot{\mathbf{B}}(\mathbf{H}_P(\tau))\cdot\mathbf{H}_P(\tau)+\tilde{\mathbf{J}}((\mathbf{E}_P*\overline{\mathbf{E}}_2)^{t+\tau})\cdot\mathbf{E}_P(\tau)]d\tau \tag{5.2}$$

because of (3.9) – (3.10) and $(3.15)_2$.

If we now suppose that two integrated histories $\overline{\mathbf{E}}_i^t$ $(i=1,2)$ are w-equivalent, then (5.1) must be satisfied whatever may be the process P and its duration d. Taking into account the expression (4.17) of the work, together with (3.9) – (3.10), which give $\mathbf{E}_P$

and $\mathbf{H}_P$ in function of the initial values $(\mathbf{E}_i(t), \mathbf{H}_i(t)),\ i = 1, 2,$ and the same process $(\dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P),$ we derive from (5.1) a relation in which the process and its duration are arbitrary. This arbitrariness yields

$$\mathbf{E}_1(t) = \mathbf{E}_2(t), \quad \mathbf{H}_1(t) = \mathbf{H}_2(t), \quad \mathbf{I}(\tau, \overline{\mathbf{E}}_1^t) = \mathbf{I}(\tau, \overline{\mathbf{E}}_2^t). \tag{5.3}$$

The last of these equalities is expressed by (4.15) and implies

$$\int_0^{+\infty} \alpha'(\tau + \xi)[\overline{\mathbf{E}}_1^t(\xi) - \overline{\mathbf{E}}_2^t(\xi)]\, d\xi = \mathbf{0}, \tag{5.4}$$

namely, the difference $\overline{\mathbf{E}}^t = \overline{\mathbf{E}}_1^t - \bar{\mathbf{E}}_2^t$ satisfies (3.17), which assures the equivalence of the two integrated histories $\overline{\mathbf{E}}_i^t$ $(i = 1, 2)$.

6 Formulation of the Maximum Recoverable Work

The maximum recoverable work expresses the maximum work we can obtain from the material at the given state σ, that is the amount of energy which is available at σ. It is defined as follows [19].

Definition 6.1 Let σ be a given state of the body $\mathcal{B}$, the maximum recoverable work starting from σ is

$$W_R(\sigma) = \sup\{-W(\sigma, P) : \ P \in \Pi\}, \tag{6.1}$$

where Π denotes the set of finite work processes.

Since the null process belongs to Π and yields a null work, $W_R(\sigma)$ is nonnegative and is bounded from above, i.e. $W_R(\sigma) < +\infty,$ as a consequence of the thermodynamics. The work defined by (6.1) has been shown to coincide with the minimum free energy, that is denoted by $\psi_m(\sigma);$ thus, we have [11, 12, 19]

$$\psi_m(\sigma) = W_R(\sigma). \tag{6.2}$$

We want to find an expression for the maximum recoverable work and hence for the minimum free energy $\psi_m(\sigma)$. For this purpose we consider as initial state $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), \overline{\mathbf{E}}^t)$ at a fixed time t and we apply a process $P \in \Pi,$ which is related to $\mathbf{E}_P$ and $\mathbf{H}_P$ by means of (3.9)–(3.10). Let d be the finite duration of P, we define P on $\mathbb{R}^+$ with its extension on $[d, +\infty),$ where we assume $P = 0$ together with $\mathbf{E}_P(d) = \mathbf{0}$ and $\mathbf{H}_P(d) = \mathbf{0}$. The work done on such a process is given by (4.17), which reduces to

$$\begin{aligned} W(\sigma, P) = &-\frac{1}{2}[\varepsilon\mathbf{E}^2(t) + \mu\mathbf{H}^2(t)] \\ &+ \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} \alpha(|\tau - \eta|)\mathbf{E}_P(\eta)\cdot\mathbf{E}_P(\tau)\, d\eta d\tau - \int_0^{+\infty} \mathbf{I}(\tau, \overline{\mathbf{E}}^t)\cdot\mathbf{E}_P(\tau)\, d\tau. \end{aligned} \tag{6.3}$$

To determine the maximum of $-W(\sigma, P)$, expressed by (6.1), we consider the set of processes which are related to

$$\mathbf{E}_P(\tau) = \mathbf{E}^{(m)}(\tau) + \gamma \mathbf{e}(\tau), \quad \tau \in \mathbb{R}^+, \tag{6.4}$$

where $\mathbf{E}^{(m)}(\tau)$ is related to the process which yields the required maximum recoverable work, γ is a real parameter and $\mathbf{e}$ is an arbitrary smooth function with $\mathbf{e}(0) = \mathbf{0}$.

From (6.3) and (6.4), we get

$$\frac{d}{d\gamma}[-W(\sigma, P)]|_{\gamma=0} = -\int_0^{+\infty} \left[\int_0^{+\infty} \alpha(|\tau-\eta|)\mathbf{E}^{(m)}(\eta)\, d\eta - \mathbf{I}(\tau, \overline{\mathbf{E}}^t) \right] \cdot \mathbf{e}(\tau)\, d\tau = 0 \tag{6.5}$$

and hence, being $\mathbf{e}$ arbitrary, we obtain

$$\int_0^{+\infty} \alpha(|\tau-\eta|)\mathbf{E}^{(m)}(\eta)\, d\eta = \mathbf{I}(\tau, \overline{\mathbf{E}}^t) \quad \forall\, \tau \in \mathbb{R}^+. \tag{6.6}$$

This relation is a the Wiener-Hopf equation of the first type, which is not solvable in general. However, by using theorems of factorization and the thermodynamic properties of the kernel α, the solution $\mathbf{E}^{(m)}$ of (6.6) can be determined and gives the maximum recoverable work [5, 6, 20], which, taking account of (6.3) and (6.6), becomes

$$W_R(\sigma) = \frac{1}{2}[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)] + \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} \alpha(|\tau-\eta|)\mathbf{E}^{(m)}(\eta) \cdot \mathbf{E}^{(m)}(\tau)\, d\eta\, d\tau. \tag{6.7}$$

Application of the Plancherel theorem allows us to transform (6.7) as follows

$$W_R(\sigma) = \frac{1}{2}[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)] + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \alpha_c(\omega)\mathbf{E}_+^{(m)}(\omega) \cdot (\mathbf{E}_+^{(m)}(\omega))^* d\omega. \tag{6.8}$$

To solve the Wiener–Hopf equation (6.6), we write it as follows

$$\int_{-\infty}^{+\infty} \alpha(|\tau-\eta|)\mathbf{E}^{(m)}(\eta)\, d\eta = \mathbf{I}(\tau, \overline{\mathbf{E}}^t) + \mathbf{r}(\tau), \quad \forall\, \tau \in R, \tag{6.9}$$

where we have added the function $\mathbf{r}$ defined by

$$\mathbf{r}(\tau) = \int_{-\infty}^{+\infty} \alpha(|\tau-s|)\mathbf{E}^{(m)}(s)\, ds \quad \forall\, \tau \in \mathbb{R}^- \tag{6.10}$$

and supposed equal to zero on R^{++}. Therefore, $supp(\mathbf{E}^{(m)}) \subseteq \mathbb{R}^+$, $supp\,(\mathbf{I}(\cdot, \overline{\mathbf{E}}^t)) \subseteq \mathbb{R}^+$, $supp\,(\mathbf{r}) \subseteq \mathbb{R}^-$ and hence (6.9) reduces to (6.6) for $\tau \in \mathbb{R}^+$ and to (6.10) for $\tau \in \mathbb{R}^-$.

In (6.7) we have the cosine Fourier transform $\alpha_c(\omega)$, which can be factorized as

$$\alpha_c(\omega) = \alpha_{(+)}(\omega)\alpha_{(-)}(\omega). \tag{6.11}$$

By introducing

$$K(\omega) = (1+\omega^2)\alpha_c(\omega), \tag{6.12}$$

which, taking into account the properties (2.18)-(2.19), is a function without zeros for every $\omega \in R$, also at infinity, and can be factorized as

$$K(\omega) = K_{(+)}(\omega)K_{(-)}(\omega), \tag{6.13}$$

we have

$$\alpha_{(+)}(\omega) = \frac{1}{1+i\omega}K_{(+)}(\omega), \qquad \alpha_{(-)}(\omega) = \frac{1}{1-i\omega}K_{(-)}(\omega). \tag{6.14}$$

Therefore, taking the Fourier transform of (6.9),

$$2\alpha_c(\omega)\mathbf{E}_+^{(m)}(\omega) = \mathbf{I}_+(\omega, \overline{\mathbf{E}}^t) + \mathbf{r}_-(\omega), \tag{6.15}$$

where $\mathbf{I}_+(\omega, \overline{\mathbf{E}}^t)$ is given by (4.16), we can evaluate the following quantity

$$\alpha_{(+)}(\omega)\mathbf{E}_+^{(m)}(\omega) = \frac{1}{2}\left[\frac{\mathbf{I}_+(\omega, \overline{\mathbf{E}}^t)}{\alpha_{(-)}(\omega)} + \frac{\mathbf{r}_-(\omega)}{\alpha_{(-)}(\omega)}\right], \tag{6.16}$$

which appears in (6.8).

Let us define

$$\mathbf{P}^t(z) = \frac{1}{4\pi i}\int_{-\infty}^{+\infty} \frac{\mathbf{I}_+(\omega, \overline{\mathbf{E}}^t)/\alpha_{(-)}(\omega)}{\omega - z}\, d\omega, \qquad \mathbf{P}^t_{(\pm)}(\omega) = \lim_{\beta\to 0^\mp} \mathbf{P}^t(\omega + i\beta); \tag{6.17}$$

hence we see that the function $\mathbf{P}^t_{(\pm)}(z)$ has zeros and singularities in $\mathbf{C}^\pm$, which implies that $\mathbf{P}^t(z) = \mathbf{P}^t_{(\pm)}(z)$ is analytic in $\mathbf{C}^{(\mp)}$ and also in R because of the hypothesis assumed for the Fourier-transformed functions on R [12]. Applying the Plemelj formulae, (6.17) yield

$$\frac{1}{2}\,\frac{\mathbf{I}_+(\omega, \overline{\mathbf{E}}^t)}{\alpha_{(-)}(\omega)} = \mathbf{P}^t_{(-)}(\omega) - \mathbf{P}^t_{(+)}(\omega), \tag{6.18}$$

which can be substituted into (6.16) to obtain the following equality

$$\alpha_{(+)}(\omega)\mathbf{E}_+^{(m)}(\omega) + \mathbf{P}^t_{(+)}(\omega) = \mathbf{P}^t_{(-)}(\omega) + \frac{1}{2}\,\frac{\mathbf{r}_-(\omega)}{\alpha_{(-)}(\omega)}. \tag{6.19}$$

In this relation the quantity at the left-hand side considered as a function of z, i.e. $\alpha_{(+)}(z)\mathbf{E}_+^{(m)}(z) + \mathbf{P}^t_{(+)}(z)$, is analytic for $z \in \mathbf{C}^-$, whereas from the right-hand side we see that $\mathbf{P}^t_{(-)}(z) + \dfrac{1}{2}\dfrac{\mathbf{r}_-(z)}{\alpha_{(-)}(z)}$ is analytic for $z \in \mathbf{C}^+$. Moreover, denoting by $\mathbf{U}(\omega)$ the function at the left-hand side of (6.19), such a function has an analytic extension on the whole complex plane and vanishes at infinity; therefore, it must be zero so that from (6.19) we get

$$\mathbf{E}_+^{(m)}(\omega) = -\frac{\mathbf{P}_{(+)}^t(\omega)}{\alpha_{(+)}(\omega)}, \qquad \mathbf{P}_{(-)}^t(\omega) = -\frac{1}{2}\frac{\mathbf{r}_-(\omega)}{\alpha_{(-)}(\omega)}. \tag{6.20}$$

From (6.20), we can evaluate $\mathbf{E}_+^{(m)}(\omega) \cdot \left(\mathbf{E}_+^{(m)}(\omega)\right)^*$, which allows to obtain the required expression of the minimum free energy

$$\psi_m(\sigma(t)) = \hat{\psi}(\mathbf{E}(t), \mathbf{H}(t), \mathbf{P}_{(+)}^t) = \frac{1}{2}[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)] + \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\mathbf{P}_{(+)}^t(\omega)|^2 d\omega. \tag{6.21}$$

7 Another Formulation for ψ_m

A different but equivalent form of the minimum free energy can be derived by expressing $\mathbf{P}_{(+)}^t(\omega)$ in terms of the integrated history $\overline{\mathbf{E}}^t$.

For this purpose we shall extend the memory kernel α' on R^{--} with an odd function $\alpha'^{(o)}(s)$, such that $\alpha'^{(o)}(s) = \alpha'(s)\ \forall s \geq 0$ and $\alpha'^{(o)}(s) = -\alpha'(-s)\ \forall s < 0$, whose Fourier transform is therefore given by (2.11), i.e. $\alpha_F'^{(o)}(\omega) = -2i\alpha_s'(\omega)$; moreover, we consider the causal extension to R^{--} for $\overline{\mathbf{E}}^t$, that is $\overline{\mathbf{E}}^t(s) = \mathbf{0}$ for all $s < 0$.

With these assumptions, (4.15) can be put in this form

$$\mathbf{I}(\tau, \overline{\mathbf{E}}^t) = \int_{-\infty}^{+\infty} \alpha'^{(o)}(\tau + \xi)\overline{\mathbf{E}}^t(\xi)\, d\xi, \quad \tau \geq 0, \tag{7.1}$$

and can be extended to R by means of

$$\mathbf{I}^{(N)}(\tau, \overline{\mathbf{E}}^t) = \int_{-\infty}^{+\infty} \alpha'^{(o)}(\tau + \xi)\overline{\mathbf{E}}^t(\xi)\, d\xi, \quad \tau < 0, \tag{7.2}$$

as follows

$$\mathbf{I}^{(R)}(\tau, \overline{\mathbb{E}}^t) = \int_{-\infty}^{+\infty} \alpha'^{(o)}(\tau + \xi)\overline{\mathbf{E}}^t(\xi)\, d\xi = \begin{cases} \mathbf{I}(\tau, \overline{\mathbf{E}}^t) & \forall \tau \geq 0, \\ \mathbf{I}^{(N)}(\tau, \overline{\mathbf{E}}^t) & \forall \tau < 0. \end{cases} \tag{7.3}$$

Let us introduce $\overline{\mathbf{E}}_N^t(s) = \overline{\mathbf{E}}^t(-s)\ \forall s \leq 0$ with its extension $\mathbf{E}_N^t(s) = \mathbf{0}\ \forall s > 0$; thus, (7.3) becomes

$$\mathbf{I}^{(R)}(\tau, \overline{\mathbf{E}}^t) = \int_{-\infty}^{+\infty} \alpha'^{(o)}(\tau - s)\overline{\mathbf{E}}_N^t(s)\, ds \tag{7.4}$$

and it gives

$$\mathbf{I}_F^{(R)}(\omega, \overline{\mathbf{E}}^t) = -2i\alpha_s'(\omega)\left(\overline{\mathbf{E}}_+^t(\omega)\right)^*, \tag{7.5}$$

since

$$\overline{\mathbf{E}}^t_{N_F}(\omega) = \overline{\mathbf{E}}^t_{N_-}(\omega) = \left(\overline{\mathbf{E}}^t_+(\omega)\right)^*. \tag{7.6}$$

From (7.3) we get

$$\mathbf{I}_F^{(R)}(\omega, \overline{\mathbf{E}}^t) = \int_{-\infty}^{+\infty} \mathbf{I}^{(R)}(\tau, \overline{\mathbf{E}}^t) e^{-i\omega\tau} d\tau = \mathbf{I}_-^{(N)}(\omega, \overline{\mathbf{E}}^t) + \mathbf{I}_+(\omega, \overline{\mathbf{E}}^t) \tag{7.7}$$

and hence

$$\frac{1}{2\alpha_{(-)}(\omega)} \mathbf{I}_F^{(R)}(\omega, \overline{\mathbf{E}}^t) = \frac{1}{2\alpha_{(-)}(\omega)} \mathbf{I}_-^{(N)}(\omega, \overline{\mathbf{E}}^t) + \mathbf{P}^t_{(-)}(\omega) - \mathbf{P}^t_{(+)}(\omega) \tag{7.8}$$

on the basis of (6.18).

Use of the Plamelj formulae yields

$$\frac{1}{2\alpha_{(-)}(\omega)} \mathbf{I}_F^{(R)}(\omega, \overline{\mathbf{E}}^t) = \mathbf{P}^{(1)\,t}_{(-)}(\omega) - \mathbf{P}^{(1)\,t}_{(+)}(\omega), \tag{7.9}$$

where $\mathbf{P}^{(1)\,t}_{(\pm)}(\omega)$ are defined as in (6.17) and, when they are considered as functions of $z \in \mathbf{C}$, have zeros and singularities in $\mathbf{C}^{\pm}$.

By substituting (7.9) into (7.8), we arrive at the function

$$\mathbf{V}(\omega) \equiv \mathbf{P}^t_{(+)}(\omega) - \mathbf{P}^{(1)\,t}_{(+)}(\omega) = \mathbf{P}^t_{(-)}(\omega) - \mathbf{P}^{(1)\,t}_{(-)}(\omega) + \frac{1}{2\alpha_{(-)}(\omega)} \mathbf{I}_-^{(N)}(\omega, \overline{\mathbf{E}}^t), \tag{7.10}$$

which is analytic in $\mathbf{C}^-$ for its first expression and in $\mathbf{C}^+$ for the second one, moreover, it vanishes at infinity. Thus, we must have $\mathbf{V}(\omega) = \mathbf{0}$ and hence

$$\mathbf{P}^t_{(+)}(\omega) = \mathbf{P}^{(1)\,t}_{(+)}(\omega), \quad \mathbf{P}^t_{(-)}(\omega) = \mathbf{P}^{(1)\,t}_{(-)}(\omega) - \frac{1}{2\alpha_{(-)}(\omega)} \mathbf{I}_-^{(N)}(\omega, \overline{\mathbf{E}}^t). \tag{7.11}$$

From (7.5), taking into account $(2.19)_1$ and (6.11), we get

$$\frac{1}{2\alpha_{(-)}(\omega)} \mathbf{I}_F^{(R)}(\omega, \overline{\mathbf{E}}^t) = i\omega\alpha_{(+)}(\omega)\big(\overline{\mathbf{E}}^t_+(\omega)\big)^*, \tag{7.12}$$

which can be substituted into the relation (6.17) written for $\mathbf{P}^{(1)\,t}_{(+)}(\omega)$ to obtain from $(7.11)_1$

$$\mathbf{P}^t_{(+)}(\omega) = \mathbf{P}^{(1)\,t}_{(+)}(\omega) = \lim_{z \to \omega^-} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega'\alpha_{(+)}(\omega')\big(\overline{\mathbf{E}}^t_+(\omega')\big)^*}{\omega' - z} \, d\omega'. \tag{7.13}$$

The conjugate of (7.13) yields

$$\big(\mathbf{P}^t_{(+)}(\omega)\big)^* = i \lim_{w \to \omega^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\omega'\alpha_{(-)}(\omega')\overline{\mathbf{E}}^t_+(\omega')}{\omega' - w} \, d\omega' \tag{7.14}$$

where, applying the Plamelj formulae, we have

$$\omega\alpha_{(-)}(\omega)\overline{\mathbf{E}}^t_+(\omega) = \mathbf{Q}^t_{(-)}(\omega) - \mathbf{Q}^t_{(+)}(\omega) \tag{7.15}$$

with

$$\mathbf{Q}^t_{(\pm)}(\omega) = \lim_{z\to\omega^\mp} \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\omega'\alpha_{(-)}(\omega')\overline{\mathbf{E}}^t_+(\omega')}{\omega' - z}\, d\omega' \tag{7.16}$$

such that $\mathbf{Q}^t_{(\pm)}(z)$ has zeros and singularities for $z \in \mathbf{C}^\pm$.

Finally, (7.14) and (7.16) give

$$\big(\mathbf{P}^t_{(+)}(\omega)\big)^* = i\mathbf{Q}^t_{(-)}(\omega), \tag{7.17}$$

which allows us to transform (6.21) as follows

$$\psi_m(t) = \hat{\psi}(\mathbf{E}(t),\, \mathbf{H}(t),\, \mathbf{Q}^t_{(-)}) = \frac{1}{2}\big[\varepsilon\mathbf{E}^2(t) + \mu\mathbf{H}^2(t)\big] + \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\mathbf{Q}^t_{(-)}(\omega)|^2 d\omega. \tag{7.18}$$

We observe that the current density given by (3.5) may be written in terms of the quantities now deduced. In fact, applying the Plancherel theorem to (3.5) and taking account of (2.11) for $\alpha'_F(\omega)$, since α' is extended to R as an odd function, and $(2.19)_1$, we have

$$\tilde{\mathbf{J}}(\overline{\mathbf{E}}^t) = -\frac{i}{\pi}\int_{-\infty}^{+\infty} \omega\alpha_c(\omega)\overline{\mathbf{E}}^t_+(\omega)\, d\omega = -\frac{i}{\pi}\int_{-\infty}^{+\infty} \alpha_{(+)}(\omega)\big[\mathbf{Q}^t_{(-)}(\omega) - \mathbf{Q}^t_{(+)}(\omega)\big] d\omega, \tag{7.19}$$

where we have considered (6.11) and (7.15) too.

In $(7.19)_2$ we can consider two integrals, one of which is zero because of the analyticity of the integrand $\alpha_{(+)}(\omega)\mathbf{Q}^t_{(+)}(\omega)$ in $\mathbf{C}^-$; thus, it follows that the other integral must be real and gives the current density, that is

$$\tilde{\mathbf{J}}(\overline{\mathbf{E}}^t) = -\frac{i}{\pi}\int_{-\infty}^{+\infty} \alpha_{(+)}(\omega)\mathbf{Q}^t_{(-)}(\omega)\, d\omega. \tag{7.20}$$

8 A Particular Model

The discrete spectrum model is characterized by a particular class of response functions, which are expressed by a linear combination of exponentials. Thus we assume the following kernel

$$\alpha(t) = \begin{cases} \sum_{i=1}^{n} g_i e^{-\alpha_i t}, & t \geq 0, \\ 0 & t < 0, \end{cases} \tag{8.1}$$

with $g_i,\ \alpha_i \in R^{++}$ $(i = 1, 2, \dots, n)$, $n \in \mathbf{N}$ and $\alpha_1 < \alpha_2 < \dots < \alpha_n$.

We first observe that with these assumptions $(2.18)_1$ is satisfied, being

$$\alpha(0) = \sum_{i=1}^{n} g_i > 0.$$

We have

$$\alpha_F(\omega) = \sum_{i=1}^{n} \frac{g_i}{\alpha_i + i\omega} \quad \forall\, \omega \in R, \tag{8.2}$$

whence

$$\alpha_c(\omega) = \sum_{i=1}^{n} \frac{\alpha_i g_i}{\alpha_i^2 + \omega^2}, \qquad \alpha_s(\omega) = \omega \sum_{i=1}^{n} \frac{g_i}{\alpha_i^2 + \omega^2} \quad \forall\, \omega \in R. \tag{8.3}$$

Then, (6.12) yields

$$K(\omega) = (1 + \omega^2) \sum_{i=1}^{n} \frac{\alpha_i g_i}{\alpha_i^2 + \omega^2}, \qquad K_\infty = \lim_{\omega \to \pm\infty} K(\omega) = \sum_{i=1}^{n} \alpha_i g_i > 0. \tag{8.4}$$

The expression $(8.4)_1$ coincides with the one obtained in [14]. Thus, we recall that, putting $z = -\omega^2$, we have $K(\omega) = f(z)$, which is a function with n simple poles at α_i^2 if $\alpha_i^2 \neq 1\ \forall\, i \in \{1, 2, \dots, n\}$, while it has $n-1$ simple poles if one of α_i's, and only one since α_i's are ordered and distinct numbers, is equal to 1.

Let $n \neq 1$.

We first suppose that $\alpha_i^2 \neq 1$ $(i = 1, 2, \dots, n)$.

If $1 < \alpha_1^2$ or $\alpha_n^2 < 1$, $f(z)$ has n simple poles α_i^2 $(i = 1, 2, \dots, n)$ and has n simple zeros denoted by $\gamma_1^2 = 1$ and γ_j^2 $(j = 2, 3, \dots, n)$. It rests to consider the case when there exists p such that $\alpha_p^2 < 1 < \alpha_{p+1}^2$, where p may assume only one of the values: $1, 2, \dots, n-1$; in this case we have the zero γ_{p+1}^2 such that $\gamma_{p+1}^2 \gtreqless 1 = \gamma_1^2$ and therefore it can be equal to 1, which becomes a zero of multiplicity 2; hence $f(z)$ has $n-1$ distinct zeros. In any case the zeros denoted by γ_j^2 $(j = 2, 3, \dots, n)$ are so ordered

$$\alpha_1^2 < \gamma_2^2 < \alpha_2^2 < \dots < \alpha_p^2 < \gamma_{p+1}^2 < \alpha_{p+1}^2 < \dots < \alpha_{n-1}^2 < \gamma_n^2 < \alpha_n^2 \tag{8.5}$$

and we can write (8.4) as follows

$$K(\omega) = K_\infty \prod_{i=1}^{n} \left\{ \frac{\gamma_i^2 + \omega^2}{\alpha_i^2 + \omega^2} \right\}, \tag{8.6}$$

where $\gamma_1^2 = 1$ and only one of the other zeros, say γ_{p+1}^2, can be equal to 1, which thus becomes a zero with multiplicity 2. Hence, it follows that the factorization (6.13) of $K(\omega)$ expressed by (8.6) yields, as in [14],

$$K_{(-)}(\omega) = k_\infty \prod_{i=1}^{n} \left\{ \frac{\omega + i\gamma_i}{\omega + i\alpha_i} \right\}, \quad K_{(+)}(\omega) = k_\infty \prod_{i=1}^{n} \left\{ \frac{\omega - i\gamma_i}{\omega - i\alpha_i} \right\}, \quad k_\infty = (K_\infty)^{1/2}. \tag{8.7}$$

Since in (7.18) we have the integral of $|\mathbf{Q}^t_{(-)}(\omega)|^2$, we must consider (7.16), where $\alpha_{(-)}(\omega)$, contrary to what occurs in [14], is now multiplied by ω. Thus, from $(6.14)_2$, taking account of $(8.7)_1$, we obtain the required product

$$\omega\alpha_{(-)}(\omega) = \omega \frac{ik_\infty}{\omega + i} \prod_{i=1}^{n} \left\{ \frac{\omega + i\gamma_i}{\omega + i\alpha_i} \right\}, \tag{8.8}$$

which also vanishes at $\gamma_0 = 0$.

Hence, since $\gamma_1 = 1$, putting $\delta_1 = \gamma_0 = 0$ and $\delta_j = \gamma_j$ $(j = 2, 3, \ldots, n)$, we get

$$\omega\alpha_{(-)}(\omega) = ik_\infty \prod_{i=1}^{n} \left\{ \frac{\omega + i\delta_i}{\omega + i\alpha_i} \right\} = ik_\infty \left(1 + i \sum_{r=1}^{n} \frac{A_r}{\omega + i\alpha_r} \right), \tag{8.9}$$

whose coefficients A_r are given by

$$A_r = (\delta_r - \alpha_r) \prod_{i=1, i\neq r}^{n} \left\{ \frac{\delta_i - \alpha_r}{\alpha_i - \alpha_r} \right\}, \quad r = 1, 2, \ldots, n, \tag{8.10}$$

where it may occur that there exists p such that $\delta_{p+1} = \gamma_{p+1} = 1$.

Then, we suppose that $\alpha_p^2 \equiv \gamma_1^2 = 1$, where p may assume only one of the values: $1, 2, \ldots, n$.

We can distinguish three cases. If $\alpha_1^2 = 1$ or $\alpha_n^2 = 1$, $f(z) = K(\omega)$ has $n-1$ zeros γ_j^2 $(j = 2, 3, \ldots, n)$ and $n-1$ poles α_j^2 $(j = 2, 3, \ldots, n)$ or α_i^2 $(i = 1, 2, \ldots, n-1)$ in the two cases; however, the presence of the factor $\dfrac{\omega}{\omega + i}$ in (8.8) still now yields the introduction of $\delta_1 = 0$ and hence (8.9) with (8.10), where $\alpha_1 = 1$ or $\alpha_n = 1$, hold again. Also in the third case with $\alpha_p^2 = 1$, $1 < p < n$, we have $n-1$ zeros γ_j^2 $(j = 2, 3, \ldots, n)$ and $n-1$ poles α_i^2 $(i = 1, 2, \ldots, p-1, p+1, \ldots, n)$ ordered as (8.5) shows with this condition $\alpha_{p-1}^2 < \gamma_p^2 < 1 < \gamma_{p+1}^2 < \alpha_{p+1}^2$; moreover, as before we get (8.9)–(8.10), where we have $\alpha_p = 1$.

Let $n = 1$.

This case must be studied separately. Substituting $(8.7)_1$, written with $n = 1$, into $(6.14)_2$, we have the following relation

$$\omega\alpha_{(-)}(\omega) = ik_\infty \frac{\omega}{\omega + i\alpha_1} = ik_\infty \left(1 + i \frac{A_1}{\omega + i\alpha_1} \right), \tag{8.11}$$

where

$$A_1 = -\alpha_1, \quad k_\infty = (\alpha_1 g_1)^{1/2}. \tag{8.12}$$

In the general case, with $n \neq 1$, $\mathbf{Q}^t_{(-)}(\omega)$, taking into account its definition (7.16) and the expression $(8.9)_2$, is given by

$$\mathbf{Q}^t_{(-)}(\omega) = ik_\infty \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\overline{\mathbf{E}}^t_+(\omega')}{\omega' - \omega^+}\, d\omega' - \sum_{r=1}^{n} k_\infty A_r \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\overline{\mathbf{E}}^t_+(\omega')/(\omega' - \omega^+)}{\omega' - (-i\alpha_r)}\, d\omega'. \quad (8.13)$$

Since $\overline{\mathbf{E}}^t_+$ as function of $z \in \mathbf{C}$ is analytic in $\mathbf{C}^{(-)}$, the first integral over the real axis can be extended to an infinite contour on $\mathbf{C}^{(-)}$ without altering its value, which is zero because of the analyticity of the integrand. It rests to evaluate the other integrals by closing again on $\mathbf{C}^{(-)}$ and taking account of the sense of the integrations. Thus, we get

$$\mathbf{Q}^t_{(-)}(\omega) = -k_\infty \sum_{r=1}^{n} \frac{A_r}{\omega + i\alpha_r} \overline{\mathbf{E}}^t_+(-i\alpha_r), \quad (8.14)$$

and hence

$$\left(\mathbf{Q}^t_{(-)}(\omega)\right)^* = -k_\infty \sum_{r=1}^{n} \frac{A_r}{\omega - i\alpha_r} \left(\overline{\mathbf{E}}^t_+(-i\alpha_r)\right)^*, \quad (8.15)$$

where, on the basis of $(2.7)_1$, we have

$$\overline{\mathbf{E}}^t_+(-i\alpha_r) = \int_0^{+\infty} e^{-\alpha_r s} \overline{\mathbf{E}}^t(s)\, ds = \left(\overline{\mathbf{E}}^t_+(-i\alpha_r)\right)^*. \quad (8.16)$$

We can now evaluate the integral

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{+\infty} &|\mathbf{Q}^t_{(-)}(\omega)|^2 d\omega \\ &= k_\infty^2 \sum_{r,l=1}^{n} A_r A_l \overline{\mathbf{E}}^t_+(-i\alpha_r) \cdot \overline{\mathbf{E}}^t_+(-i\alpha_l) \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i/(\omega + i\alpha_r)}{\omega - i\alpha_l} d\omega \quad (8.17) \\ &= K_\infty \sum_{r,l=1}^{n} \frac{A_r A_l}{\alpha_r + \alpha_l} \overline{\mathbf{E}}^t_+(-i\alpha_r) \cdot \overline{\mathbf{E}}^t_+(-i\alpha_l), \end{aligned}$$

which, substituted into (7.18), by virtue of $(8.16)_1$, yields

$$\begin{aligned} \psi_m(t) &= \frac{1}{2}[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)] \\ &+ \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} 2K_\infty \sum_{r,l=1}^{n} \frac{A_r A_l}{\alpha_r + \alpha_l} e^{-(\alpha_r s_1 + \alpha_l s_2)} \overline{\mathbf{E}}^t(s_1) \cdot \overline{\mathbf{E}}^t(s_2)\, ds_1 ds_2. \end{aligned} \quad (8.18)$$

In the particular case, when $n = 1$, taking account of (8.12), the expression of the minimum free energy assumes the simpler form

$$\psi_m(t) = \frac{1}{2}\left[\varepsilon \mathbf{E}^2(t) + \mu \mathbf{H}^2(t)\right] + \frac{1}{2}\alpha_1^2 g_1 \left[\int_0^{+\infty} e^{-\alpha_1 s} \overline{\mathbf{E}}^t(s)\, ds\right]^2. \quad (8.19)$$

We observe that, integrating by parts in this last relation, we obtain the same result derived in [14], where the history of **E** is considered instead of its integrated history.

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