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Asymptotic Behavior and Stability of the Solutions of Functional Differential Equations in Hilbert Space

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Abstract: In the following article we present the results on the asymptotic behavior and stability of the strong solutions for functional differential equations (FDE). We also formulate several results on spectral properties (completeness and basisness) of exponential solutions of the above-mentioned equations. It is relevant to underline that our approach for researching FDE is based on the spectral analysis of the operator pencils which are the symbols (characteristic quasipolynomials) with operator coefficients. The article is divided into two parts. The first part is devoted to researching FDE in a Hilbert space, the second part to researching FDE in a finite-dimensional space.

Keywords: *Functional-differential equation; solvability; stability; operator-valued function; asymptotic behavior; basisness.*

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1 Introduction

In the first part of this article we present results on the unique solubility of initial-boundary-value problems for a certain class of linear difference-differential equation of neutral type with coefficients that are operator-valued functions taking values in a set of operators (in general, unbounded) in a Hilbert space. We consider the case of variable

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time-lag. Moreover, we establish results for asymptotic behavior and the stability of the solutions of the above-mentioned equations.

These results extend certain results obtained in [1 − 3, 35 − 38, 40 − 43].

In the second part of this article we study the asymptotic behavior of the solutions of difference-differential equations (in finite-dimensional space $H = C^m$) in more complicated and delicate situations in which there are the chains of roots of characteristic quasipolynomials which are lying on or approaching the imaginary axis (so-called critical and supercritical cases). There are several results, devoted to the analysis of this situation (see for more details [19 − 22]).

Besides, it is relevant to underline that our approach is seriously different to those methods used in cited works. Our estimates of the solutions are based on Riesz basis property of the system of exponential solutions. In turn, this result is based on the researching of the resolvent of the generator of the C^0-semigroup of the shift operator naturally connected with the initial-value problem for difference-differential equation.

These results generalize certain results obtained in [4, 35 − 37, 39].

It is relevant to note that at the end of Section 1 and of Section 2 we give references for and brief comments on a comparison of our results with the results of previously published works on the subject.

2 FDE in Infinite Dimensional Space

Let H be a separable Hilbert space, let A be a positive self-adjoint operator in H with a bounded inverse, and let I be the identity operator in H. We convert the domain $\mathrm{Dom}(A^\alpha)$ of operator A^α $(\alpha > 0)$ into a Hilbert space H_α by introducing the norm $\|\cdot\|_\alpha = \|A^\alpha \cdot\|$ on $\mathrm{Dom}(A^\alpha)$.

We denote by $W_2^1((a,b),A)$ $(-\infty < a < b \le +\infty)$ the space of functions with values in H such that $A^j v^{(1-j)}(t) \in L_2((a,b),H)$ $(j = 0,1)$, endowed with the norm

$$\|v\|_{W_2^1(a,b)} \equiv \left(\int_a^b \left(\|v^{(1)}(t)\|^2 + \|Av(t)\|^2\right) dt \right)^{\frac{1}{2}}.$$

Here and throughout $v^{(j)}(t) \equiv \frac{d^j}{dt^j} v(t)$, $j = 0, 1, \ldots$. See Chapter 1 in [5] for more detailed information and a description of the space $W_2^1((a,b),A)$.

Along with $W_2^1((a,b),A)$ we introduce the two spaces $L_{2,\gamma}((a,b),H)$ and $W_{2,\gamma}^1((a,b),A)$ of functions with values in H, with norms defined by the relations

$$\|v\|_{L_{2,\gamma}} \equiv \left(\int_a^b \exp(-2\gamma t)\|v(t)\|^2 \, dt \right)^{\frac{1}{2}},$$

$$\|v\|_{W_{2,\gamma}^1(a,b)} \equiv \|\exp(-\gamma t) v(t)\|_{W_2^1(a,b)}, \quad \gamma \in \mathbb{R}.$$

We consider the following problem on the semiaxis $\mathbb{R}_+ = (0, +\infty)$

$$\begin{aligned} \mathcal{U}u \equiv \frac{du}{dt} + Au(t) + B_0(t)CAu(t) \\ + \sum_{j=1}^{n} \left(B_j(t) S_{g_j}(Au)(t) + D_j(t) S_{g_j}\left(\frac{du}{dt}\right)(t) \right) = f(t), \end{aligned} \tag{1}$$

$$u(+0) = \phi_0. \tag{2}$$

Here $B_0(t)$, $B_j(t)$ and $D_j(t)$ $(j = 1,2,\dots,n)$ are strongly continuous (see [6]) operator-valued functions with values in the ring of bounded operators in the space H, C is a compact operator in H, and $\phi_0 \in H_{\frac{1}{2}}$.

We define the operators S_{g_j} as follows

$$(S_{g_j}v)(t) = v(g_j(t)), \quad g_j(t) \ge 0,$$
$$(S_{g_j}v)(t) = 0, \quad g_j(t) < 0, \quad j = 0,1,2,\dots,n,$$

where $g_j(t)$ $(j = 1,2,\dots,n)$ are real-valued functions with continuous derivatives on the semiaxis $\mathbb{R}_+$ such that $g_j(t) \le t$; $\frac{d}{dt}g_j(t) > 0$ $(j = 1,2,\dots,n)$, and $g_0(t) = t$, $t \in \mathbb{R}_+$. We shall denote by $g_j^{-1}(t)$ the inverse functions of $g_j(t)$, $h_j(t) = t - g_j(t)$.

Definition 2.1 We call a *vector-valued function* $u(t)$ a *strong solution of equation* (1) if this function is in the space $W^1_{2,\gamma}(\mathbb{R}_+, A)$ for some value of $\gamma \ge 0$ and satisfies (1) almost everywhere on $\mathbb{R}_+$.

We define the quantities

$$r_1(\gamma) = \sup_{\lambda:\ \Re\lambda > \gamma} \|A(\lambda I + A)^{-1}\|,$$
$$r_2(\gamma) = \sup_{\lambda:\ \Re\lambda > \gamma} |\lambda|\, \|(\lambda I + A)^{-1}\|, \quad \gamma \ge 0.$$

Theorem 2.1 *Let us suppose that* $B_0(t) \equiv 0$ *and there exists* γ_0 *such that*

$$\sigma(\gamma_0) < 1, \tag{3}$$

where

$$\sigma(\gamma) = r_1(\gamma) \sum_{j=1}^{n} \sup_{t \in [g_j^{-1}(0),+\infty)} \left[\exp\left(-\gamma(t - g_j(t))\right) \|B_j(t)\| \left(\frac{1}{g_j^{(1)}(t)} \right)^{\frac{1}{2}} \right]$$
$$+ r_2(\gamma) \sum_{j=1}^{n} \sup_{t \in [g_j^{-1}(0),+\infty)} \left[\exp\left(-\gamma(t - g_j(t))\right) \|D_j(t)\| \left(\frac{1}{g_j^{(1)}(t)} \right)^{\frac{1}{2}} \right].$$

Then for every $\gamma > \gamma_0$ *the operator* V_γ*, acting according to the rule* $V_\gamma u \equiv (\mathcal{U}u, u(+0))$*, takes the space* $W^1_{2,\gamma}(\mathbb{R}_+, A)$ *onto* $L_{2,\gamma}(\mathbb{R}_+, H) \oplus H_{\frac{1}{2}}$ *and has a bounded inverse.*

We now turn to the problem often called the initial-value problem:

$$\frac{du}{dt} + Au(t) + B_0(t)CAu(t) + \sum_{j=1}^{n} \left(B_j(t)Au(g_j(t)) + D_j(t)u^{(1)}(g_j(t))\right) = f_0(t), \quad t \in \mathbb{R}_+, \tag{1°}$$

$$u^{(m)}(t) = y_m(t), \quad t \in \mathbb{R}_- = (-\infty, 0), \quad u(+0) = \phi_0, \quad m = 0,1. \tag{2°}$$

It is known (see, for details Chapter 1 in [8]) that problem (1°), (2°) can be reduced to one of the form (1), (2). In this case the vector-valued function $f(t)$ is defined as follows:

$$f(t) = f_0(t) - \sum_{j=1}^{n} \left[B_j(t)T^{g_j}(Ay_0)(t) + D_j(t)T^{g_j}(y_1)(t) \right], \tag{4}$$

where the operators T^{g_j} are defined in a following way

$$(T^{g_j}v)(t) = 0, \quad g_j(t) \ge 0, \qquad (T^{g_j}v)(t) = v(g_j(t)), \quad g_j(t) < 0.$$

Definition 2.2 We call a *vector-valued function* $u(t)$ belonging to the space $W^1_{2,\gamma}(\mathbb{R}_+, A)$ for some $\gamma \geq 0$ the *strong solution of the problem* (1°), (2°), if $u(t)$ satisfies equation (1) with the function $f(t)$ defined by the expression (4) and the condition (2) in the sense of convergence in the space $H_{\frac{1}{2}}$.

On the basis of Theorem 2.1 it is possible to obtain

Theorem 2.2 *Let us suppose that the conditions of Theorem 2.1 are satisfied and there exists* $\gamma_1 > 0$*, such that*

$$\sigma_1(\gamma_1) < +\infty, \tag{5}$$

$$\sigma_1(\gamma) = \sum_{j=1}^{n} \sup_{t\in[0,g_j^{-1}(0))} \left[\exp\left(-\gamma(t-g_j(t))\right) \|B_j(t)\| \left(\frac{1}{g_j^{(1)}(t)} \right)^{\frac{1}{2}} \right]$$
$$+ \sum_{j=1}^{n} \sup_{t\in[0,g_j^{-1}(0))} \left[\exp\left(-\gamma(t-g_j(t))\right) \|D_j(t)\| \left(\frac{1}{g_j^{(1)}(t)} \right)^{\frac{1}{2}} \right].$$

Then for every $\gamma \geq \gamma_* = \max(\gamma_0, \gamma_1)$*, every vector-valued functions* $(Ay_0)(t)$*,* $y_1(t) \in L_{2,\gamma}(\mathbb{R}_-, H)$*,* $f(t) \in L_{2,\gamma}(\mathbb{R}_+, H)$ *and every vector* $\phi_0 \in H_{\frac{1}{2}}$ *there exists a unique solution* $u(t)$ *of the problem* (1°), (2°) *belonging to the space* $W^1_{2,\gamma}(\mathbb{R}_+, A)$ *and satisfying the inequality*

$$\|u(t)\|_{W^1_{2,\gamma}(\mathbb{R}_+,A)} \leq d_1 \Big(\|f_0(t)\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|Ay_0\|^2_{L_{2,\gamma}(\mathbb{R}_-,H)}$$
$$+ \|y_1\|^2_{L_{2,\gamma}(\mathbb{R}_-,H)} + \|\phi_0\|^2_{\frac{1}{2}} \Big)^{\frac{1}{2}} \tag{6}$$

with constant d *independent of* $(f_0(t), (Ay_0)(t), y_1(t), \phi_0)$.

In the following theorem we investigate the case $\gamma_0 = 0$ which is important in applications.

Theorem 2.3 *Let us suppose that* $B_0(t) \equiv 0$ *and the following inequality*

$$\sum_{j=1}^{n} \left[\varlimsup_{t\to+\infty} \left(\|B_j(t)\|^2 \frac{1}{g_j^{(1)}(t)} \right)^{\frac{1}{2}} + \varlimsup_{t\to+\infty} \left(\|D_j(t)\|^2 \frac{1}{g_j^{(1)}(t)} \right)^{\frac{1}{2}} \right] < 1 \tag{7}$$

satisfies.

Then the conclusion of Theorem 2.1 holds with constant $\gamma_0 = 0$ *and for* $\gamma_0 = 0$ *and* $f(t) \in L_2(\mathbb{R}_+, H)$

$$\lim_{t\to+\infty} \|u(t)\|_{\frac{1}{2}} = 0.$$

The following statement is connected with the equation of retarded type $(D_j(t) \equiv 0$, $j = 1, 2, \ldots, n)$.

Theorem 2.4 *Let us suppose* $D_j(t) \equiv 0$*,* $j = 1, 2, \ldots, n$*, operator-valued functions* $B_j(t)$ *are represented by the expression* $B_j(t) = B_j^0(t)C_j$*, where* C_j *are the compact* *operators in the space H, $B_j^0(t)$ — are strongly continuous operator-valued functions taking values in the ring of bounded operators in H and such that*

$$\sup_{t\in\mathbb{R}_+}\|B_0(t)\| < +\infty, \qquad \sup_{t\in\mathbb{R}_+}\left(\|B_j^0(t)C_j\|^2\,\frac{1}{g_j^{(1)}(t)}\right)^{\frac{1}{2}} < +\infty. \tag{8}$$

Then there exists $\gamma_0 \ge 0$ such that for every $\gamma \ge \gamma_0$ the operator V_γ takes the space $W^1_{2,\gamma}(\mathbb{R}_+, A)$ onto $L_{2,\gamma}(\mathbb{R}_+, H) \oplus H_{\frac{1}{2}}$ and has a bounded inverse.

Let us denote by α_0 the infimum of the operator A (see the definition in [6]).

Theorem 2.5 is devoted to the case of negative γ_0.

Theorem 2.5 *Let us suppose the conditions of Theorem 2.4 are satisfied, $B_0(t) \equiv 0$, the inequality*

$$\sum_{j=1}^{n}\left(\sup_{t\in[g_j^{-1}(0),+\infty)}\|B_j(t)C_j\|^2\,\frac{1}{g_j^{(1)}(t)}\right)^{\frac{1}{2}} < 1 \tag{9}$$

holds and the delays $h_j(t)$ are bounded: $0 < \theta_1 \le h_j(t) \le \theta_2 < \infty$; $\theta_1, \theta_2 = \mathrm{const}$.

Then there exists $\delta > 0$ such that for every $\gamma > \max(-\delta, -\alpha)$ the operator V_γ takes the space onto the $L_{2,\gamma}(\mathbb{R}_+, H) \oplus H_{\frac{1}{2}}$ and has a bounded inverse.

We present the result which is the corollary of Theorem 2.5.

Theorem 2.6 *Let us suppose the conditions of Theorem 2.5 are satisfied, $f_0(t) \equiv 0$, and the inequality*

$$\omega_0 = \max_{j=\overline{1,\,n}}\ \sup_{t\in[0,g_j^{-1}(0))}|g_j(t)| < +\infty$$

holds.

Then there exists $\delta > 0$ such that for every initial functions $y_0(t)$, $y_1(t)$ such that $Ay_0(t)$, $y_1(t) \in L_2((-\omega_0, 0), H)$ and every vector $\phi_0 \in H_{\frac{1}{2}}$ there exists the unique solution $u(t)$ of the problem (1°), (2°) (for $f_0 \equiv 0$), belonging to the space $W^1_{2,\gamma}(\mathbb{R}_+, A)$ (for $\gamma > \max(-\delta, -\alpha_0)$) and satisfying the inequality

$$\|e^{-\gamma t}u(t)\|_{W_2^1(\mathbb{R}_+,A)} \le d_2\left(\|\phi_0\|^2_{\frac{1}{2}} + \|Ay_0\|^2_{L_2(-\omega_0,0)} + \|y_1\|^2_{L_2(-\omega_0,0)}\right)^{\frac{1}{2}} \tag{10}$$

with the constant d_2 independent of $(\phi_0, (Ay_0)(t), y_1(t))$.

Let us present several remarks connected with the conditions of the results that were formulated above.

Remark 2.1 Under the additional restriction $h_j(t) \ge \alpha > 0$, $t \ge 0$, $j = 1, 2, \dots, n$ the sufficient condition for the existence γ_0 in the inequality (3) is the following

$$\begin{aligned}\Delta \equiv \sum_{j=1}^{n}\Bigg[&\sup_{t\in[g_j^{-1}(0),+\infty)}\left(\|B_j(t)\|^2\,\frac{1}{g_j^{(1)}(t)}\right)^{\frac{1}{2}} \\ &+ \sup_{t\in[g_j^{-1}(0),+\infty)}\left(\|D_j(t)\|^2\,\frac{1}{g_j^{(1)}(t)}\right)^{\frac{1}{2}}\Bigg] < +\infty.\end{aligned} \tag{11}$$

Remark 2.2 If the inequality (11) holds and the delays $h_j(t) \geq \alpha > 0$, then the inequality (3) holds for every γ_0 such that

$$\gamma_0 > \frac{1}{\alpha}\max(\ln\Delta, 0). \tag{12}$$

Remark 2.3 The condition (5) guarantees that the right-hand part of (4) belongs to the space $L_{2,\gamma}(\mathbb{R}_+, H)$ for every vector-valued functions $(Ay_0)(t),\ y_1(t) \in L_{2,\gamma}(\mathbb{R}_-, H)$.

Moreover, if the functions $g_j(t)$ satisfy the condition

$$\omega_0 = \max_{j=\overline{1,n}} \sup_{t\in[0, g_j^{-1}(0))} |g_j(t)| < +\infty,$$

then the second part of (4) belongs to the space $L_{2,\gamma}(\mathbb{R}_+, H)$ for every $\gamma \in \mathbb{R}$ and every vector-valued functions $y_0(t)$, $y_1(t)$ such that $(Ay_0)(t),\ y_1(t) \in L_2((-\omega_0, 0), H)$.

Remark 2.4 The inequality (7) is essential. It may be demonstrated by the following example.

Example 2.1 Let us suppose $H \equiv \mathbf{C},\ n = 1,\ A = \text{const} > 0,\ B_1(t) \equiv -1,\ B_0(t) \equiv D_1(t) \equiv 0$. Let us consider the following problem

$$\frac{du}{dt} + Au(t) - S_{t-h}(Au)(t) = \chi(0,h)A, \quad t \in \mathbb{R}_+, \tag{E1}$$
$$u(+0) = \phi_0 = 1,$$

where $\chi(0,h)$ is a characteristic function of the interval $(0,h)$.

In this case the condition (7) is not satisfied (the left-hand side of (7) is equal to 1). Equation (E1) has a unique solution $u(t) \equiv 1$ which does not belong to the space $W_2^1(\mathbb{R}_+)$.

Remark 2.5 The condition (11) is also essential. It may be demonstrated by the following example.

Example 2.2 Let us suppose $H = C,\ n = 1,\ B_0(t) \equiv D_1(t) \equiv 0,\ g_1(t) = t - 1,\ A = \text{const} > 0,\ B_1(t) = -2t\exp(2t - (A+1))$. Let us consider the following problem

$$\frac{du}{dt} + Au + B_1(t)u(t-1) = 0, \quad t \in \mathbb{R}_+, \tag{E2}$$
$$u(t) = y(t) = \exp(t^2 - At), \quad t \in [-1, 0], \quad u(+0) = \phi_0 = 1.$$

In this case equation (E2) has the solution $u(t) = \exp(t^2 - At)$ which does not belong to the space $W_{2,\gamma}^1(\mathbb{R}_+)$ for any $\gamma \in \mathbb{R}$.

Owing to the fact that

$$\sup_{t\in[1,+\infty)} \|B_1(t)\| = +\infty,$$

the condition (11) is not satisfied.

Remark 2.6 The inequality (12) is essential. It may be demonstrated by the following example.

Example 2.3 Let us suppose $H = C,\ n = 1,\ B_0(t) \equiv B_1(t) \equiv 0,\ A = \text{const} > 0$, $D_1(t) \equiv D = \text{const},\ h > 0$. Let us consider the following equation

$$\frac{du}{dt} + Au(t) + D\frac{du}{dt}(t-h) = 0, \quad t \in \mathbb{R}_+. \tag{E3}$$

In this case the roots of the associated quasipolynomial

$$l(\lambda) = \lambda + A + \lambda D e^{-\lambda h}$$

are asymptotically approaching the line $\Re\lambda = \frac{\ln|D|}{h} = \Delta$ (see, for example [7]), being on the left-hand side if

$$\Re\lambda_q < \frac{\ln|D|}{h}.$$

Thus it is impossible to change Δ by $\Delta - \varepsilon$ (for any $\varepsilon > 0$) in the inequality (12).

The papers [30 − 33] were devoted to the spectral problem, namely, to studying operator-valued functions, which are symbols of the considered equations in the autonomous case.

Now we are going to present certain results for the asymptotic behavior of the strong solutions of FDE in the autonomous case. These results are based on information on the symbols (characteristic quasipolynomials) of the above-mentioned equations. In turn, these symbols are operator-valued functions (operator pencils) taking values in a set of unbounded operators in a Hilbert space.

The papers [30 − 33] dealt with operator-valued functions of the form

$$\begin{aligned} \text{Ł}(\lambda) = \lambda I + A + B_0 C A + \sum_{j=1}^{n}(B_j A + \lambda D_j)\exp(-\lambda h_j) + \\ + \left(\int_0^\infty \exp(-\lambda t)K(t)\,dt\right)A + \lambda\left(\int_0^\infty \exp(-\lambda t)Q(t)\,dt\right). \end{aligned} \tag{13}$$

Here B_0, B_j and D_j are bounded operators in the space H, $0 = h_0 < h_1 < \cdots < h_n = h$, the operator functions $e^{-\varkappa t}K(t)$ and $e^{-\varkappa t}Q(t)$ take values in the ring of bounded operators acting in the space H and such that the operator functions $e^{-\varkappa t}K(t)$ and $e^{-\varkappa t}Q(t)$ are Bochner integrable on the semiaxis $\mathbb{R}_+$ for some $\varkappa \geq 0$, and λ $(\lambda \in \mathbb{C})$ is a spectral parameter.

A number of papers were devoted to studying characteristic quasipolynomials, the distribution of its zeros, and its estimates in the case of a finite-dimensional space H. We only mention monographs [7, 9, 10] and papers [19, 29].

The operator-valued functions of the form (13) have been studied much less in the case of infinite-dimensional spaces and, in particular, a Hilbert space H. Moreover, we do not know any paper (except [17, 18]) specifically dedicated to studying operator functions of the form (13).

It is noteworthy that there are new unexpected phenomena in the case of infinite-dimensional spaces. Some illustrative examples were given in [30, 32].

Let us proceed by formulating the certain results from [30 − 33, 35].

Lemma 2.1 *Let B_0, B_j, and D_j $(j = 1, 2, \ldots, n)$ be bounded operators in the space H, let the operator functions $K(t)$ and $Q(t)$ take values in the ring of bounded operators, acting in the space H, let the operator functions $\exp(-\varkappa t)K(t)$ and $\exp(-\varkappa t)Q(t)$ be Bochner integrable on the semiaxis $\mathbb{R}_+$ for some $\varkappa \geq 0$. Then there exists $M_0 \geq \varkappa$ such that in the half-plane $\Pi(M_0) \equiv \{\lambda\colon \Re\lambda > M_0\}$ the operator function $L^{-1}(\lambda)$ exists, is holomorphic, and satisfies the inequality*

$$\|(L(\lambda)(\lambda I + A)^{-1})^{-1}\| \leq \text{const}\,. \tag{E4}$$

The following lemma defines conditions for the meromorphity of the operator-valued function $Ł^{-1}(\lambda)$.

Lemma 2.2 *Let the assumptions of Lemma 2.1 be satisfied. In addition, suppose that B_j $(j = 1, 2, \ldots, n)$ are compact operators in the space H, the operator functions $K(t)$ and $Q(t)$ take values in the ring of compact operators on H and additionally satisfy the condition $K(t) = Q(t) = 0$ whereas $t > h \stackrel{\text{def}}{=} h_n$. Then the spectrum of $L^{-1}(\lambda)$ consists of isolated characteristic numbers of a finite algebraic multiplicity that are finite-dimensional poles of $L^{-1}(\lambda)$.*

The following two statements complement Lemma 2.1 in the case of delay equations, i.e. $D_j \equiv 0$, $j = 1, 2, \ldots, n$, $Q(t) \equiv 0$.

Lemma 2.3 *Let the assumptions of Lemma 2.2 be satisfied and let $D_j \equiv 0$, $j = 1, 2, \ldots, n$, and $Q(t) \equiv 0$. Then for any $a \geq 0$ there exists $b > 0$ such that in the domain*

$$Q(a, b) \equiv \mathbb{C}\backslash\big(\{\lambda\colon \Re\lambda \leq -a\} \cup \{\lambda\colon -a \leq \Re\lambda \leq M_0,\ |\Im\lambda| \leq b\}\big),$$

the operator-valued function $L^{-1}(\lambda)$ exists, is holomorphic, and satisfies inequality (E4).

Lemma 2.4 *Let the assumptions of Lemma 2.2 be satisfied, let $D_j = 0$, $j = 1, 2, \ldots, n$, let $Q(t) \equiv 0$, let the operators B_j can be represented in the form $B_j = C_j A^{-\theta_j}$ $(j = 1, 2, \ldots, n)$, where C_j are bounded operators in the space H, $\theta_j \in (0, 1]$, $K(s) = K_1(s)A^{-\theta_0}$, where $\theta_0 \in (0, 1]$, and the operator function $K_1(s)$ takes values in the ring of bounded operators in the space H and is Bochner integrable on the interval $(0, h)$. Then there exists a constant $N_0 > 0$ such that in the domain*

$$\Phi(N_0) \equiv \mathbb{C}\backslash\big(\{\lambda\colon |\lambda| \leq N_0\} \cup \{\lambda\colon \Re\lambda < 0,\ |\Im\lambda| \leq N_0 \exp(-q\Re\lambda)\}\big),$$

where

$$q = \max\left(\max_{j=1,n} \frac{h_j}{\theta_j},\ \frac{h}{\theta_0}\right),$$

the operator-valued function $L^{-1}(\lambda)$ exists, is holomorphic, and satisfies inequality (E4).

Here it is useful to note the following.

Remark 2.7 Under the assumptions of Lemma 2.3, the assertion of Lemma 2.1 is valid for any constant $M_0 > \max \lambda_q$, where by λ_q we denote characteristic numbers of the operator-valued function $Ł(\lambda)$.

The major parts of papers [3, 30, 33, 35] are devoted to studying a more specific case of problem (1), (2) related to autonomous equation (1) with $f_0(t) \equiv 0$.

We assume that $B_0(t) \equiv B_0$, $B_j(t) \equiv B_j$, and $D_j(t) \equiv D_j$ and the functions $h_j(t) \equiv h_j$ $(j = 1, 2, \dots, n)$ are independent of t, i.e. B_0, B_j, and D_j are bounded operators in the space H, the operator functions $K(t)$ and $Q(t)$ satisfy the assumptions of Lemma 2.1, and h_j are numbers such that $0 = h_0 < h_1 < \dots < h_n = h$.

For convenience we formulate the resultant problem:

$$\frac{du}{dt} + Au(t) + B_0 CAu(t) + \sum_{j=1}^{n} \big(B_j Au(t-h_j) + D_j u^{(1)}(t-h_j)\big) + \int_{-\infty}^{t} \big(K(t-s)Au(s) + Q(t-s)u^{(1)}(s)\big)\,ds = 0, \quad t \in \mathbb{R}_+, \tag{$1^{\circ\circ}$}$$

$$u^{(m)}(t) = y_m(t), \quad t \in \mathbb{R}_- = (-\infty, 0), \quad m = 0, 1; \qquad u(+0) = \varphi_0. \tag{$2^{\circ\circ}$}$$

Following the lines of [24], we introduce the operators $\mathbf{F}_1$ and $\mathbf{F}_2$, acting in the space $L_2((-h, 0), H)$:

$$(\mathbf{F}_1 v)(t) = -\sum_{j=1}^{n} \chi(-h_j, 0)(t) B_j v(-t-h_j) - \int_{-h}^{t} K(-s)v(s-t)\,ds,$$

$$(\mathbf{F}_2 v)(t) = -\sum_{j=1}^{n} \chi(-h_j, 0)(t) D_j v(-t-h_j) - \int_{-h}^{t} Q(-s)v(s-t)\,ds,$$

$$t \in [-h, 0),$$

where $\chi(-h_j, 0)(t)$ are characteristic functions of the intervals $(-h_j, 0)$.

The next statement is useful when studying spectral problems.

Assertion 2.1 *Let B_0, B_j, D_j $(j = 1, 2, \dots, n)$ be bounded operators in the space H, and let the operator functions $K(t)$ and $Q(t)$ satisfy the assumptions of Lemma 2.2. Then any strong solution $u(t)$ of problem $(1^{\circ\circ})$, $(2^{\circ\circ})$ satisfies the inequalities*

$$d_1 \|u\|_{W_2^1(0,h)} \le \big(\|\varphi\|_{1/2}^2 + \|\mathbf{F}_1(Ay_0)(t) + \mathbf{F}_2(y_1)(t)\|_{L_2(-h,0)}^2\big)^{1/2} \le d_2 \|u\|_{W_2^1(0,h)}$$

with constants d_1 and d_2 independent of $(\varphi_0, \mathbf{F}_1(Ay_0), \mathbf{F}_2(y_1))$.

By U_α we denote the set of strong solutions of equation $(1^{\circ\circ})$ such that $\exp(\alpha t)u(t) \in L_2(\mathbb{R}_+, H)$, $\alpha \in \mathbb{R}$.

On the base of the canonical system of eigenvectors and adjoint eigenvectors $x_{q,j,0}$, $x_{q,j,1}, \dots, x_{q,j,s}$ $(j = 1, 2, \dots, p_q,\ s = 0, 1, \dots, r_{pq})$ of the operator-valued function $\mathrm{L}(\lambda)$ we construct the system of elementary (exponential) solutions of equation $(1^{\circ\circ})$:

$$y_{q,j,s}(t) = \exp(\lambda_q t)\bigg(\frac{t^s}{s!}\, x_{q,j,0} + \frac{t^{s-1}}{(s-1)!}\, x_{q,j,1} + \dots + x_{q,j,s}\bigg).$$

Lemma 2.5 *Let $D_j = 0$ $(j = 1, 2, \ldots, n)$, let $Q(t) \equiv 0$, let B_j $(j = 1, 2, \ldots, n)$ be compact operators in the space H, let the operator function $K(t)$ take values in the ring of compact operators acting in the space H, and let $K(t) = 0$, $t > h$. Then for an arbitrary $\alpha \geq 0$ any strong solution $u(t)$ of problem $(1^{\circ\circ})$, $(2^{\circ\circ})$ can be expressed in the form*

$$u(t) = \sum_{\Re\lambda_q \geq -\alpha} \sum_{j=1}^{p_q} \sum_{s=0}^{r_{pq}} c_{q,j,s} y_{q,j,s}(t) + w_\alpha(t),$$

where the vector-valued function $w_\alpha(t)$ belongs to U_α, and the coefficients $c_{q,j,s}$ satisfy the inequalities

$$|c_{q,j,s}| \leq d_q \big(\|\varphi_0\|_{1/2}^2 + \|\mathbf{F}_1(Ay_0)(t)\|_{L_2(-h,0)}^2\big)^{1/2}$$

with the constants d_q independent of $(\varphi_0, \mathbf{F}_1(Ay_0)(t))$.

Corollary 2.1 *Let the conditions of Lemma 2.5 be satisfied, and let the solution $u(t)$ belong to U_α. Then there exists $\delta > 0$ such that $u(t) \in U_{\alpha+\delta}$.*

Lemma 2.6 *Let B_0, B_j, and D_j $(j = 1, 2, \ldots, n)$ be bounded operators in the space H, let $K(t)$ and $Q(t)$ be operator functions taking values in the ring of bounded operators on the space H and such that $\exp(-\varkappa t)K(t)$ and $\exp(-\varkappa t)Q(t)$ are Bochner integrable on the semiaxis $\mathbb{R}_+$ for some $\varkappa \geq 0$. Then the assertion of Theorem 2.1 is valid for any constant $\gamma_0 = M_0$, where the constant M_0 is defined in Lemma 2.1.*

Lemma 2.7 *Let the assumptions of Lemma 2.5 with $\varkappa = 0$ be valid, and let the operator function $(Ł(\lambda)(\lambda I + A)^{-1})^{-1}$ be bounded and continuous in the operator norm on the imaginary axis and satisfy the inequality*

$$\sup_{\lambda:\ \Re\lambda \geq 0} \|Ł(\lambda)(\lambda I + A)^{-1} - I\| < 1.$$

Then the assertion of Theorem 2.1 is valid with the constant $\gamma_0 = 0$; moreover, any solution of the problem with $\gamma = \gamma_0 = 0$ and $f(t) \in L_2(\mathbb{R}_+, H)$ satisfies the relation

$$\lim_{t \to +\infty} \|u(t)\|_{1/2} = 0.$$

Remark 2.8 We point out that the proposed approach to definition and understanding the solution of the problem (1°), (2°) is by no means the only one possible.

To date there is an extensive literature (covering mainly the case of finite-dimensional space) where one can find various approaches to the interpretation of solutions and various methods of solution and analysis of initial-boundary-value problems for functional differential equations. Here we restrict ourselves to drawing attention to monographs and papers [7−12] devoted to this subject and papers [13−17] treating the case of equations in Banach and, in particular, in Hilbert spaces.

Our approach to the interpretation of the solutions of FDE in a Hilbert space is based on the approach presented in [8, 11], and it is its development to FDE in abstract spaces.

We remark that the results that were formulated (Theorems 2.1−2.6) may be obtained in the same way as Theorem 1 [1] and Theorem 1 [3]. Certain differences arise in estimates of the integral operators in the description of the integral equation which is equivalent in the sense of solvability of the problem (1), (2).

We point out that, although there are many papers devoted to the study of functional differential equations in a Banach space, in particular, in Hilbert space, they consider mainly delay-type equations. We know of far fewer papers considering abstract equations of neutral type. The papers closest to the subject of present work are $[14-15,18]$.

In the papers that we know ($[14-15,18]$) the restrictions on the coefficients $B_j(t)A$ and $D_j(t)$ $(j=1,2,\dots,n)$ of the delays are more severe. For example, in most papers (see, in particular, $[14-15]$) the authors assume that the coefficients of the delays ($B_j(t)A$ and $D_j(t)$) are bounded operators. The authors of $[13,16-17]$ assume in the case of delay-type equations $(D_j(t)\equiv 0,\ j=1,2,\dots,n)$ that the coefficients $B_j(t)$ are independent of t.

It is relevant to underline that in articles $[42-46]$ we obtained results on Fredholm solubility and the properties of the strong solutions of FDE of n-th order of convolution type (including integro-differential equations) the symbols of which are operator-valued functions representable as operator bundle of n-th order perturbed by operator-valued functions of special type (bounded or decreasing at infinity). In $[44-48]$ we proved also the result about multiple minimality of the system root vectors and exponential solutions.

In turn in the papers $[30-33,35,40,46,47]$ we proved the results on asymptotic behavior of the strong solutions of FDE in a Hilbert space, and in particular, the results on the nonexistence of nontrivial solutions decreasing more rapidly than any exponent (the problem of so-called small solutions or the Phraghmen–Lindelöf principle).

3 FDE in Finite-Dimensional Space

We are going to study the asymptotic behavior of the solutions of the following equation

$$\sum_{j=0}^{n}\left(B_j u(t-h_j)+D_j\,\frac{du}{dt}\,(t-h_j)\right)+\int_0^h B(s)u(t-s)\,ds=0,\qquad t\in\mathbb{R}_+. \tag{14}$$

Here B_j, D_j $(j=0,1,\dots,n)$ are $(m\times m)$ matrices with constant elements, the real numbers h_j satisfy the inequalities $0=h_0<h_1<\dots<h_n=h$, the elements $B_{ij}(s)$ of the matrix $B(s)$ belong to the space $L_2((0,h),C)$.

Let us introduce the matrix-valued function

$$\mathcal{L}(\lambda)=\sum_{j=0}^{n}(B_j+\lambda D_j)\exp(-\lambda h_j)+\int_0^h \exp(-\lambda s)B(s)\,ds,\qquad \lambda\in C, \tag{15}$$

and the complex-valued function $l(\lambda)=\det\mathcal{L}(\lambda)$ often called by the characteristic quasipolynomial of equation (14).

Let us denote by λ_q the zeroes of the function $l(\lambda)$ numbered in increasing order of its modulars (counting multiplicity).

The eigenvectors appearing in a canonical system of eigen and associated (root) vectors corresponding to λ_q we denote by $x_{q,j,0}$, and associated vector of order s by $x_{q,j,s}$ (the index j shows where is vector $x_{q,j,0}$ in a sequence of the vectors in specially chosen basis of subspace of solutions of the equation $\mathcal{L}(\lambda_q)x=0$).

We introduce the system of exponential (elementary) solutions of equation (14)

$$y_{q,j,s}(t)=\exp\left(\lambda_q t\right)\left(\frac{t^s}{s!}\,x_{q,j,0}+\frac{t^{s-1}}{(s-1)!}\,x_{q,j,1}+\dots+x_{q,j,s}\right). \tag{16}$$

Let us denote by $W_2^1((a,b),C^m)$ $(-\infty < a < b \le +\infty)$ the Sobolev space of functions with values in C^m endowed by the norm

$$\|v\|_{W_2^1(a,b)} \equiv \left(\int_a^b \left(\|v^{(1)}(t)\|_{C^m}^2 + \|v(t)\|_{C^m}^2 \right) dt \right)^{\frac{1}{2}}.$$

Along with $W_2^1((a,b),C^m)$ we introduce $W_{2,\gamma}^1((a,b),C^m)$ as the space of functions with values in C^m endowed by the norm

$$\|v\|_{W_{2,\gamma}^1(a,b)} \equiv \left(\int_a^b e^{-2\gamma t} \left(\|v^{(1)}(t)\|_{C^m}^2 + \|v(t)\|_{C^m}^2 \right) dt \right)^{\frac{1}{2}}, \qquad \gamma \in \mathbb{R}.$$

We state for equation (14) the following initial conditions

$$u(t) = y(t), \quad t \in [-h, 0], \quad u(+0) = y(-0), \\ y(t) \in W_2^1((-h,0),C^m). \tag{17}$$

Definition 3.1 We call a *vector-valued function* $u(t)$ belonging to the space $W_{2,\gamma}^1((-h,+\infty),C^m)$ for certain $\gamma \in \mathbb{R}$ a *strong solution of the problem* (14), (17), if $u(t)$ satisfies equation (14) almost everywhere on the semiaxis $\mathbb{R}_+$ and condition (17).

First of all we formulate an a priori estimate for the strong solutions of the problem (14), (17).

Lemma 3.1 *Let us suppose* $\det D_0 \neq 0$. *Then there exists* $\gamma_0 \ge 0$ *such, that for every* $\gamma \ge \gamma_0$ *the problem (14), (17) has a unique strong solution* $u(t) \in W_{2,\gamma}^1((-h,+\infty), C^m)$ *for every initial function* $y(t) \in W_2^1((-h,0),C^m)$, *and this solution* $u(t)$ *satisfy the inequality*

$$\|u\|_{W_{2,\gamma}^1((-h,+\infty),C^m)} \le d\|y\|_{W_2^1((-h,0),C^m)} \tag{18}$$

with constant d *independent of function* $y(t)$.

Keeping in mind Lemma 2.1 let us introduce (in a way similar to that in [7]) the semigroup U_t of bounded operators, acting in the space $W_2^1((-h,0),C^m)$ according to the rule

$$(U_t y)(s) = u(t+s), \quad -h \le s \le 0, \quad t \ge 0.$$

Here $u(t)$ is the solution of the problem (14), (17) corresponding to the initial function $y(s)$.

In the following theorem we present the description of the generator of C^0-semigroup U_t.

Theorem 3.1 *Let us suppose* $\det D_0 \neq 0$. *Then* U_t *is* C^0*-semigroup of the operators acting in the space* $W_2^1((-h,0),C^m)$ *with generator* $\mathcal{D}$ *such that*

$$(\mathcal{D}\phi)(s) = \frac{d\phi}{ds}(s), \qquad s \in (-h,0),$$

$$\operatorname{Dom}\mathcal{D} = \Bigg\{ \phi \in W_2^2((-h,0),C^m),\ \sum_{j=0}^m \left(B_j\phi(-h_j) + D_j\phi^{(1)}(-h_j) \right) \\ + \int_0^h B(s)\phi(-s)\,ds = 0 \Bigg\}. \tag{19}$$

Proposition 3.1 *Let us suppose* $\det D_0 \neq 0$*. Then the spectrum of the operator* $\mathcal{D}$ *is the set* Λ *of the zeroes* λ_q *of the function* $l(\lambda)$ *and the exponential solutions* $y_{q,j,s}(t)$ *(see (16)) are its root vectors and form a minimal system in the space* $W_2^1((-h,0),C^m)$*.*

In the following theorem we present the result on completeness of the system of exponential solutions.

Theorem 3.2 *Let us suppose* $\det D_0 \neq 0$*,* $\det D_n \neq 0$*. Then the system of elementary solutions* $\{y_{q,j,s}(t)\}$ *is complete in the space* $W_2^1((-h,0),C^m)$*.*

In the following proposition we show the localization of the spectrum of the operator $\mathcal{D}$.

Proposition 2.2 *Let us suppose* $\det D_0 \neq 0$*,* $\det D_n \neq 0$*. Then there exist constants* α *and* β *such that the set* Λ *is lying in the strip* $\{\lambda\colon \alpha < \Re\lambda < \beta\}$*.*

Let us denote by V_{λ_q} the span of the elementary solutions $y_{q,j,s}(t)$, corresponding to λ_q, by ν_q the multiplicity of λ_q and by $\varkappa = \sup\limits_{\lambda_q\in\Lambda} \Re\lambda_q$, $N = \max\limits_{\lambda_q\in\Lambda} \nu_q$.

Now we present one of our main results on the behavior of the strong solutions of the problem (14), (17).

Theorem 3.3 *Let us suppose that* $\det D_0 \neq 0$*,* $\det D_n \neq 0$ *and the set* Λ *is separate:*

$$\inf_{\lambda_p\neq\lambda_q} (\operatorname{dist}(\lambda_p,\lambda_q)) > 0.$$

Then any strong solution $u(t)$ *of the problem (14), (17) satisfies the inequality*

$$\|u(t+\cdot)\|_{W_2^1(-h,0)} \leq d(t+1)^{N-1} \exp(\varkappa t) \|y\|_{W_2^1(-h,0)}, \quad t \geq 0, \tag{20}$$

with constant d *independent of* $y(t)$*.*

The theorem is based on the following result.

Theorem 3.4 *Let us suppose the conditions of Theorem 3.3 are satisfied.*
Then the system of subspaces V_{λ_q} $(\lambda_q \in \Lambda)$ *forms a Riesz basis of subspaces of the space* $W_2^1((-h,0),C^m)$*.*

Let $B_p(\lambda_q)$ be a disk with radius ρ and with a center at the point λ_q. We introduce the domain

$$G_\rho(\Lambda) \equiv C \setminus \bigcup_{\lambda_q\in\Lambda} B_\rho(\lambda_q).$$

Assertion 3.1 *Let* $\det D_0 \neq 0$ *and* $\det D_n \neq 0$*. Then there exists a system of closed contours* $\Gamma_n = \{\lambda\colon \Re\lambda = \beta,\ c_n \leq \Im\lambda \leq c_{n+1}\} \cup \{\lambda\colon \Re\lambda = \alpha,\ c_n \leq \Im\lambda \leq c_{n+1}\} \cup l_{n+1}$*,* $n \in \mathbb{Z}$*, which entirely lies in the domain* G_ρ *for some sufficiently small* $\rho > 0$*. In addition, the following conditions are satisfied:*

(i) *The sequence of real numbers* $\{c_n\}$ $(n \in \mathbb{Z})$ *lying on the semiaxes* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ *is such that* $0 < \delta \leq c_{n+1} - c_n \leq \Delta < +\infty$*; piecewise smooth curves* l_n*, joining the points* $(\Re\lambda = \beta, \Im\lambda = c_n)$ *and* $(\Re\lambda = \alpha, \Im\lambda = c_n)$ *do not intersect, and their lengths* $d(l_n)$ *are uniformly bounded with respect to* n *(here* δ *and* Δ *are positive constants).*

(ii) *The number* $N(\Gamma_n)$ *of zeros* λ_q *(with regard to their multiplicities) lying inside the contour* Γ_n *is uniformly bounded with respect to* n*:*

$$\max_{n\in\mathbb{Z}} N(\Gamma_n) \leq M < +\infty;$$

(iii) *There exists a constant* c *such that* $\sup\limits_{\lambda\in\Gamma_n} |\lambda|\,\|\mathcal{L}^{-1}(\lambda)\| \leq c$*.*

We introduce the set $\{\mathcal{P}_n\}$ of Riesz spectral projections, corresponding to the contours Γ_n:

$$(\mathcal{P}_n f) = -\frac{1}{2\pi i}\int\limits_{\Gamma_n} R(\lambda,\mathbb{D})f\,d\lambda, \qquad n\in\mathbb{Z},$$

in this case we assume that contours have the counterclockwise orientation.

The following Theorems 3.5 and 3.6 generalize the Theorems 3.3 and 3.4.

Theorem 3.5 *Let* $\det D_0 \neq 0$ *and* $\det D_n \neq 0$. *Then there exists a system of contours* Γ_n $(n\in\mathbb{Z})$, *satisfying the conditions* (i)$-$(iii) *of Assertion 3.1, such that the corresponding system of subspaces* $\mathcal{W}_n = \mathcal{P}_n W_2^1((-h,0),C^m)$ *forms a Riesz basis of subspaces of the space* $W_2^1((-h,0),C^m)$.

On the basis of Theorem 3.5 can be obtained the following

Theorem 3.6 *Let* $\det D_0 \neq 0$ *and* $\det D_n \neq 0$. *Then any strong solution* $u(t)$ *of the problem (14), (17) satisfies the inequality*

$$\|u(t+\cdot)\|_{W_2^1((-h,0),C^m)} \le d_1(t+1)^{M-1}\exp(\varkappa t)\|y\|_{W_2^1((-h,0),C^m)}, \qquad t\ge 0, \tag{21}$$

with the constant M *defined in Assertion 3.1 and constant* d_1 *independent of* $y(t)$.

The following theorem generalizes Theorem 3.3 in the case $B(s)\equiv 0$.

Theorem 3.7 *Let* $D_0\neq 0$, $\inf_{\lambda_p\neq\lambda_q}(\mathrm{dist}(\lambda_p,\lambda_q))>0$, $B(s)\equiv 0$.
Then any strong solution $u(t)$ *of the problem (14), (17) satisfies the inequality (20).*

It is relevant to underline that the estimate (20) is also valid for the well-known example of Gromova and Zverkin (see [20], and the remarks in the monograph [7]). In their example $m=1$, $n=1$, $D_0=-D_1=1$, $B_0=B_1=a=\mathrm{const}>0$, $N=1$, $\varkappa=0$. Moreover, if we introduce the following norm:

$$\|u\|^*_{W_2^1(-h,0)} = \left(\int\limits_{-h}^{0}\left(|u^{(1)}(s)|^2+a^2|u(s)|^2\right)ds + a\left(|u(0)|^2+|u(-h)|^2\right)\right)^{\frac12}$$

which is equivalent to the traditional norm in the space $W_2^1((-h,0),C)$ the exponential solutions $e^{\lambda_q t}$ will be orthogonal in scalar product $\langle\cdot,\cdot\rangle^*_{W_2^1}$ associated with norm $\|\cdot\|^*_{W_2^1(-h,0)}$.

In addition to the Theorems 3.3, 3.4 and we present (formulate) results on the asymptotic behavior of the strong solutions of scalar difference-differential equation of the m-th order.

Let us denote by $W^m_{2,\gamma}((a,b),C)$ weighted Sobolev space of complex-valued functions with norm

$$\|u\|_{W^m_{2,\gamma}(a,b)} = \left[\int\limits_a^b \exp(-2\gamma t)\left(\sum_{j=0}^m |u^{(j)}(t)|^2\right)dt\right]^{\frac12}, \qquad \gamma\in\mathbb{R}.$$

We study the following initial value problem:

$$\sum_{j=0}^{m}\sum_{k=0}^{n} a_{kj}u^{(j)}(t-h_k) + \int_0^h a(s)u(t-s)\,ds = 0, \qquad t \in \mathbb{R}_+; \tag{22}$$

$$\begin{gathered} u(t) = y(t), \quad t \in [-h, 0], \\ u^{(j)}(+0) = y^{(j)}(-0), \quad j = 0, 1, \dots, m-1. \end{gathered} \tag{23}$$

Here a_{kj} are the complex coefficients, real numbers h_j satisfy the inequalities $0 = h_0 < h_1 < \dots < h_n = h$, the function $a(s) \in L_2((0,h), C)$, the initial data $y(s) \in W_2^m((-h,0), C)$.

Definition 3.2 We call the *complex-valued function* $u(t)$ belonging to the space $W_{2,\gamma}^m((-h,+\infty), C)$ for some $\gamma \geq 0$ the *strong solution of the problem* (22), (23), if $u(t)$ satisfies equation (22) almost everywhere on the semiaxis $\mathbb{R}_+$ and the initial conditions (23).

Let us denote by ν_q the multiplicities of the zeroes λ_q of the function $l(\lambda)$

$$l(\lambda) = \sum_{j=0}^{m}\sum_{k=0}^{n} a_{kj}\lambda^j \exp(-\lambda h_k) + \int_0^h a(s)e^{-\lambda s}\,ds. \tag{24}$$

Theorem 3.8 *Let us suppose* $a_{0m} \neq 0$*,* $a_{nm} \neq 0$*, and the set* Λ *of all zeroes* λ_q *of the function* $l(\lambda)$ *is separate (that's* $\inf\limits_{\lambda_p \neq \lambda_q} \operatorname{dist}(\lambda_q, \lambda_p) > 0$*).*

Then the strong solution $u(t)$ *of the problem (22), (23) satisfies the inequality*

$$\|u(t+\cdot)\|_{W_2^m(-h,0)} \leq d(t+1)^{N-1} \exp(\varkappa t)\|y\|_{W_2^m(-h,0)}, \qquad t \geq 0, \tag{25}$$

with constant d *independent of* $y(t)$*. Here* $N = \max\limits_{\lambda_q \in \Lambda} \nu_q$*,* $\varkappa = \sup\limits_{\lambda_q \in \Lambda} \Re\lambda_q$*.*

This theorem is based on the following result.

Theorem 3.9 *Let us suppose that the conditions of Theorem 3.7 are satisfied.*

Then the following system of functions

$$v_{q,m} = \frac{t^r \exp(\lambda_q t)}{(|\lambda_q|^m + 1)}, \quad \lambda_q \in \Lambda, \quad r = 0, 1, \dots, \nu_q - 1; \tag{26}$$

form a Riesz basis in the space $W_2^m(-h, 0)$*.*

Remark 3.1 The inequality $a_{nm} \neq 0$ is essential for Riesz basisness. Indeed it is not difficult to verify that for the following difference-differential equation

$$\frac{du}{dt} + au(t) + bu(t-h) = 0$$

the system of normed exponential solutions $y_q(t) = a_q e^{\lambda_q t}$ $(\|y_q(t)\|_{W_2^1(-h,0)} = 1)$ is not uniformly minimal. This fact may be easily obtained by calculating the scalar product $\langle y_{q+1}(t), \overline{y}_q(t)\rangle_{W_2^1(-h,0)}$.

Using the well-known asymptotics of the zeroes λ_q of the quasipolynomial

$$l(\lambda) = \lambda + a + be^{-\lambda h}$$

one can verify that

$$\langle y_{q+1}(t), \overline{y}_q(t) \rangle_{W_2^1(-h,0)} \to 1 \quad (q \to +\infty). \tag{27}$$

The statement (27) is a contradiction of uniform minimality of the system $\{y_q(t)\}_{\lambda_q \in \Lambda}$.

Remark 3.2 It is relevant to underline that critical and supercritical cases are realized for quasipolynomials (24), when

$$|a_{0m}| = |a_{nm}|$$

(see [19, 20] for more details).

Remark 3.3 It is known that in the case $k(s) \equiv 0$ constant N satisfies the following inequality

$$N \le m(n+1) - 1.$$

It is relevant to underline that one of the first results about geometrical properties of elementary solutions of an equation similar to [22] was obtained by Levinson and McCalla in 1974 in [23]. In [23] a result on the completeness and minimality of the system of exponential solutions for the equation of the retarded type $(a_{ni} = 0,\ i = 1, 2, \ldots, n)$ was obtained.

The generalization of this result for retarded equations $D_j \equiv 0,\ j = 1, \ldots, n$ in the space $\mathbb{R}^n$ was obtained by Delfour and Manitius in [24]. In turn, the strongest results on the completeness of the exponential autonomous FDE were obtained by Lunel [25 − 27]. It is important to underline that in [25 − 27] Lunel also considered the problem of so-called "small solutions" which is deeply connected with the problem of the completeness of the exponential solutions.

The problem of small solutions was also researched by Hale [7], Henry [28] and Kappel [29] in finite-dimensional space $H = \mathbb{R}^m(C^m)$.

Certain results about minimality of the elementary solutions and the problem of small solutions (Phraghmen–Lindelöf Principle) for FDE in a Hilbert space was obtained by author in [30 − 33, 40].

In cited papers [30 − 33, 35] one can also find results on the spectral properties of the operator-valued functions (operator pencils) that are the symbols (characteristic quasipolynomials) of the autonomous FDE with operator coefficients in a Hilbert space (see also references in [30 − 33]).

Recently results on Riesz basisness in the space $L_2((-h, 0), C^m)$ of the exponential solutions for FDE of neutral type with a different understanding of solvability and definition of solutions have been obtained by Lunel and Yakubovich in [34].

For a more complete description of our results on Riesz basisness and estimates of the strong solutions presented in this article see [2 − 4, 35 − 37, 39].

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