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Stability of an Autonomous System with Quadratic Right-Hand Side in the Critical Case

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Abstract: In this paper an autonomous system of differential equations with quadratic right-hand side is considered. In the case when the matrix of linear approximation has just one zero eigenvalue, the stability of trivial solution is investigated. System is written in the vectors-matrices form and under some additional conditions a Liapunov function of the quadratic form is constructed. A guaranteed zone of stability of trivial solution is given as well.

Keywords: *Zero eigenvalue; Lyapunov stability.*

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1 Introduction

Many problems of biological sciences, medicine sciences etc. lead to investigation of systems that are described by means of ordinary differential equations with quadratic right-hand sides (e.g. [3, 5]). Zero solution of the system with quadratic right-hand side in the case of presence of zero eigenvalue of matrix of corresponding linear part can be, in general, unstable. This effect occurs already in the scalar case. For instance, the trivial solution of simple scalar equation $\dot{x} = -x^2$ is unstable, since the solution of the initial

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problem $x(t_0) = x_0$, given by formula $x(t) = 1/(t - t_0 + x_0^{-1})$ has, in the case $x_0 < 0$, a limit $\lim\limits_{t \to t_0 - x_0^{-1} - 0} x(t) = -\infty$.

We shall refer to as *critical cases* such cases when between the eigenvalues of the matrix of corresponding linear approximation there is at least one zero eigenvalue and the remaining eigenvalues have negative real parts. Then stability or instability cannot be established by the linear approximation. In the present paper one of critical cases for autonomous system with quadratic right-hand side, which allows the stability, is considered. This system consists of $n+1$ equations and has the form

$$\begin{aligned} \dot{x}_i &= \sum_{s=1}^{n} a_{is} x_s + \sum_{k,s=1}^{n} b_{ks}^i x_k x_s + 2 \sum_{k=1}^{n} b_{k,n+1}^i x_k z + b_{n+1,n+1}^i z^2, \qquad i = 1 \dots, n, \\ \dot{z} &= \sum_{k,s=1}^{n} b_{ks}^{n+1} x_k x_s + 2 \sum_{k=1}^{n} b_{k,n+1}^{n+1} x_k z + b_{n+1,n+1}^{n+1} z^2, \end{aligned} \tag{1}$$

where the coefficients a_{is} and b_{kl}^m are constant (we suppose $b_{kl}^m = b_{lk}^m$ if both coefficients exist) and it is supposed that the matrix of linear approximation has just one zero eigenvalue.

For further investigation system (1) is written in the unified vectors-matrices form. As a tool of investigation, a Liapunov function of quadratic form is used. When the full derivative of the Liapunov function along the trajectories of system (1) is estimated, the coefficients of the resulting form are chosen in such a way that guarantees its nonpositivity in a neighbourhood of zero equilibrium state.

Moreover, as a consequence of the performed computations, in the case of stability a concrete neighbourhood of zero solution is found, where fulfilling of the definition of stability is guaranteed. This is possible due to the involved vectors-matrices method. For such kinds of neighbourhoods the term *guaranteed zone of stability* was involved previously (see e.g. [7]). To the best of our knowledge there is no result (for the discussed critical case) which is considered by means of the vectors-matrices method. The estimation of guaranteed zone of stability is new as well.

In the sequel the norms, used for vectors and matrices, are defined as

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

for the vector $x = (x_1, \dots, x_n)$ and

$$\|A\| = \left(\lambda_{\max}(A^T A) \right)^{1/2}$$

for any $m \times n$-matrix A. Here and in the sequel $\lambda_{\max}(\cdot)$ (or $\lambda_{\min}(\cdot)$) is maximal (or minimal) eigenvalue of the corresponding symmetric and positive definite matrix ([6]).

2 Preliminaries

Let us consider an autonomous system with quadratic right-hand sides

$$\dot{y}_i = \sum_{s=1}^{n+1} c_s^i y_s + \sum_{k,s=1}^{n+1} d_{ks}^i y_k y_s, \qquad i = 1, \dots, n+1, \tag{2}$$

where c_s^i and d_{ks}^i with $d_{ks}^i = d_{sk}^i$, $i, k, s = 1, \dots, n+1$ are constants.

If the matrix of linear approximation of system (2), i.e. the matrix of the system

$$\dot{y}_i = \sum_{s=1}^{n+1} c_s^i y_s, \qquad i = 1, \dots, n+1,$$

has just one zero eigenvalue, then there exists a linear regular transformation of the form

$$x_i = \sum_{s=1}^{n+1} l_s^i y_s, \qquad i = 1, \dots, n,$$

$$z = \sum_{s=1}^{n+1} l_s^{n+1} y_s,$$

(where l_s^i and l_s^{n+1}, $s = 1, \dots, n+1$, $i = 1, \dots, n$ are constants and x_i, $i = 1, \dots, n$; z are new dependent variables) which transform this system to the system (1) (see e.g. [9]). Therefore the investigation of the system (1) instead of the general case of system (2) is well grounded.

We begin with some necessary definitions of stability. Let us consider the general system of differential equations

$$\dot{y} = f(t, y), \quad y \in \mathbb{R}^{n+1} \tag{3}$$

with $f\colon [t^*, \infty) \times \Omega \to \mathbb{R}^{n+1}$, where Ω is a connected domain containing the origin of coordinates and $f(t, 0) \equiv 0$ for all $t \in [t^*, \infty)$. Besides it is supposed that through each point $(t_0, y_0) \in [t^*, \infty) \times \Omega$ just one solution $y(t) = y(t; t_0, y_0)$ passes. Maximal right-hand interval of existence of this solution we denote as $J^+(t_0, y_0)$. By definition, $y(t_0; t_0, y_0) = y(t_0)$.

Definition 2.1 [8, 11] *Solution* $y \equiv 0$ of the system (3) is called *stable* if for every $\varepsilon > 0$ and every $t_0 \in [t^*, \infty)$ there exists a $\delta > 0$ such that for any $y_0 \in \mathbb{R}^{n+1}$ with $\|y_0\| < \delta$ and for any $t \in J^+(t_0, y_0)$ it follows: $\|y(t, t_0, y_0)\| < \varepsilon$.

Definition 2.2 [10, 11] *Solution* $y \equiv 0$ of the system (3) is called *uniformly stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $t_0 \in [t^*, \infty)$, $y_0 \in \mathbb{R}^{n+1}$ with $\|y_0\| < \delta$ and any $t \in J^+(t_0, y_0)$ it follows: $\|y(t, t_0, y_0)\| < \varepsilon$.

Obviously, for the autonomous systems under consideration the notions of stability and uniform stability are equivalent. Stability of system (1) will be investigated by means of the direct (second) Liapunov method. For this the following general result is necessary.

Theorem 2.1 [8, 11] *If there exist a function* $V\colon [t^*, \infty) \times \Omega \to \mathbb{R}^+$, $V \in C^1$ *and an increasing continuous function* $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ *with* $a(0) = 0$ *such that for all* $(t, y) \in [t^*, \infty) \times \Omega$*:*

(1) $V(t, y) \ge a(\|y\|)$*;* $V(t, 0) = 0$*;*

(2) $\dot{V}(t, y) \le 0$*,*

where $\dot{V}$ *is the total derivative of the function* V *along the trajectories of system (3), then the solution* $y \equiv 0$ *of this system is stable.*

Note that function V is usually called the *Liapunov* function. In addition to the establishment of the fact of stability (or asymptotic stability), the second Liapunov method gives a possibility of estimation of the domain (guaranteed zone) of stability (or asymptotic stability), i.e. gives a possibility of estimation of a set of initial data $y_0 \in \mathbb{R}^{n+1}$ for which the corresponding definitions of stability hold. The guaranteed zone of stability can be defined with the aid of the Liapunov function as a set of $y \in \mathbb{R}^{n+1}$ such that

$$V(t, y) < \alpha,$$

where $\alpha = \text{const}$, in the situation when Theorem 2.1 is valid. If this set is equal to the space $\mathbb{R}^{n+1}$ (i.e. if α can be taken as any positive number), we say that the trivial solution is *globally stable*. In particular, for linear autonomous systems stability (asymptotic stability) is always global.

If the system (3) is linear and autonomous, i.e. has the form

$$\dot{y} = Ay,$$

where A is an $n \times n$-matrix, we can look for a Liapunov function of the quadratic form

$$V(y) = y^T H y.$$

Then

$$\dot{V}(y) = y^T (A^T H + HA) y.$$

Let, moreover, A be asymptotically stable (i.e. all its eigenvalues have negative real parts). Then always there exists a symmetric positive definite $n \times n$-matrix H such that the symmetric matrix

$$C = -A^T H - HA$$

is positive definite too (see $[1, 2, 4, 8]$). The set of the matrices H, satisfying this property, generates a convex cone (on the set of positive definite matrices) and the zero matrix serves as its vertex. The corresponding function V satisfies all conditions formulated above (in Theorem 2.1).

3 Matrix Forms of System (1)

Let us consider the system (1). For investigation of stability of its trivial solution it will be useful to rewrite system (1) in the vectors-matrices form. Further we will use: matrices $X_i,\ i = 1, \dots, n$ of the type $n \times n$ and $Z_i,\ i = 1, \dots, n$ of the type $n \times 1$ having the property that only the i-th row of which can be nonzero; symmetric matrices $B_l,\ l = 1, \dots, n+1$ of the type $n \times n$; vectors $b_l,\ l = 1, \dots, n+1$ of the type $n \times 1$; matrix A of the type $n \times n$; vector θ of the type $n \times 1$; matrix Θ of the type $n \times n$ and vector x of the type $n \times 1$. They are defined according to the following formulas:

$$X_i = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad Z_i = \begin{pmatrix} 0 \\ . \\ z \\ . \\ 0 \end{pmatrix}, \quad B_l = \begin{pmatrix} b^l_{11} & b^l_{12} & \dots & b^l_{1n} \\ b^l_{21} & b^l_{22} & \dots & b^l_{2n} \\ \dots & \dots & \dots & \dots \\ b^l_{n1} & b^l_{n2} & \dots & b^l_{nn} \end{pmatrix},$$

$$b_l = \begin{pmatrix} b^l_{1,n+1} \\ b^l_{2,n+1} \\ \dots\dots \\ b^l_{n,n+1} \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \\ . \\ 0 \end{pmatrix},$$

$$\Theta = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}.$$

Then the system (1) can be rewritten in the form

$$\frac{d}{dt}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} A & \theta \\ \theta^T & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ z \end{pmatrix}$$

$$+ \begin{pmatrix} X_1 & Z_1 & \dots & X_n & Z_n & \Theta & \theta \\ \theta^T & 0 & \dots & \theta^T & 0 & x^T & z \end{pmatrix} \cdot \begin{pmatrix} B_1 & b_1 \\ b_1^T & b^1_{n+1,n+1} \\ \dots & \dots \\ B_{n+1} & b_{n+1} \\ b^T_{n+1} & b^{n+1}_{n+1,n+1} \end{pmatrix} \cdot \begin{pmatrix} x \\ z \end{pmatrix}$$

or in the form

$$\frac{d}{dt}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} A + r_{11}(x,z) & r_{12}(x,z) \\ r^T_{21}(x,z) & r_{22}(x,z) \end{pmatrix} \cdot \begin{pmatrix} x \\ z \end{pmatrix} \tag{4}$$

with

$$r_{11} = \sum_{l=1}^{n}(X_l B_l + Z_l b_l^T), \qquad r_{12} = \sum_{l=1}^{n}(X_l b_l + Z_l b^l_{n+1,n+1}),$$
$$r^T_{21} = x^T B_{n+1} + z b^T_{n+1}, \quad r_{22}(x,z) = x^T b_{n+1} + z b^{n+1}_{n+1,n+1}. \tag{5}$$

4 Main Result

Before formulation of the main result let us introduce necessary abbreviations:

$$\tilde{b}_{n+1} = \left(b^1_{n+1,n+1},\ b^2_{n+1,n+1},\ \dots,\ b^n_{n+1,n+1}\right)^T,$$

$$\tilde{B}_{n+1} = \begin{pmatrix} b^1_{1,n+1} & b^1_{2,n+1} & \dots & b^1_{n,n+1} \\ b^2_{1,n+1} & b^2_{2,n+1} & \dots & b^2_{n,n+1} \\ \dots & \dots & \dots & \dots \\ b^n_{1,n+1} & b^n_{2,n+1} & \dots & b^n_{n,n+1} \end{pmatrix}, \tag{6}$$

$$R(H) = 2(\tilde{B}^T_{n+1}H + H\tilde{B}_{n+1} + B_{n+1}),$$

where H is an $n \times n$ matrix and

$$\bar{B}^T = \begin{pmatrix} b^1_{11} & \dots & b^1_{1n} & b^1_{21} & \dots & b^1_{2n} & \dots & b^1_{n1} & \dots & b^1_{nn} \\ b^2_{11} & \dots & b^2_{1n} & b^2_{21} & \dots & b^2_{2n} & \dots & b^2_{n1} & \dots & b^2_{nn} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b^n_{11} & \dots & b^n_{1n} & b^n_{21} & \dots & b^n_{2n} & \dots & b^n_{n1} & \dots & b^n_{nn} \end{pmatrix}. \tag{7}$$

Theorem 4.1 *Let the matrix A be asymptotically stable and the coefficient $b^{n+1}_{n+1,n+1} = 0$. If there exists a symmetric positive definite $n \times n$ matrix H such that the matrix*

$$C = -A^T H - HA$$

is positive definite too and, moreover, the relation

$$H\tilde{b}_{n+1} + 2b_{n+1} = 0 \tag{8}$$

holds, then the trivial solution of system (1) is stable. A guaranteed zone of stability contains an ellipse

$$x^T Hx + z^2 \le \alpha$$

with

$$\alpha = \frac{\lambda_{\max}(H) \cdot (\lambda_{\min}(C))^2}{\lambda_{\max}(H)\|R(H)\|^2 + 4\|H\bar{B}^T\|^2}.$$

Remark 4.1 With respect to the definition of stability we note, that if conditions of Theorem 4.1 hold, then (as it follows from the proof) for each solution (x, z) of system (1) defined by the initial data (t_0, x_0, z_0) with $x_0^T Hx_0 + z_0^2 \le \alpha$ we have $J^*(t_0, x_0, z_0) = [t_0, \infty)$.

Proof of Theorem 4.1 Let us seek for a Liapunov function of the hypermatrix form

$$V(x, z) = (x^T, z) \cdot \begin{pmatrix} H & \theta \\ \theta^T & h_{n+1,n+1} \end{pmatrix} \cdot \begin{pmatrix} x \\ z \end{pmatrix},$$

where $h_{n+1,n+1}$ is a positive constant. Its total derivative along the trajectories of system (4) takes the form

$$\dot{V}(x, z) = (x^T, z) \cdot \left\{ \begin{pmatrix} A^T + r_{11}^T(x, z) & r_{21}(x, z) \\ r_{12}^T(x, z) & r_{22}(x, z) \end{pmatrix} \cdot \begin{pmatrix} H & \theta \\ \theta^T & h_{n+1,n+1} \end{pmatrix} \right.$$
$$\left. + \begin{pmatrix} H & \theta \\ \theta^T & h_{n+1,n+1} \end{pmatrix} \cdot \begin{pmatrix} A + r_{11}(x, z) & r_{12}(x, z) \\ r_{21}^T(x, z) & r_{22}(x, z) \end{pmatrix} \right\} \cdot \begin{pmatrix} x \\ z \end{pmatrix}.$$

After computing we get

$$\dot{V}(x, z) = (x^T, z) \cdot \begin{pmatrix} g_{11}(x, z) & g_{12}(x, z) \\ g_{12}^T(x, z) & g_{22}(x, z) \end{pmatrix} \cdot \begin{pmatrix} x \\ z \end{pmatrix}$$

with

$$g_{11}(x, z) = (A + r_{11}(x, z))^T H + H(A + r_{11}(x, z)),$$
$$g_{12}(x, z) = r_{21}(x, z)h_{n+1,n+1} + Hr_{12}(x, z)$$

and

$$g_{22}(x, z) = 2h_{n+1,n+1}r_{22}(x, z).$$

With the aid of (5) we express

$$g_{11}(x,z) = (A^T H + HA) + \sum_{l=1}^{n} \big((X_l B_l)^T H + H(X_l B_l)\big) + \sum_{l=1}^{n} \big((Z_l b_l^T)^T H + H(Z_l b_l^T)\big),$$

$$g_{12}(x,z) = \bigg(h_{n+1,n+1} B_{n+1} x + H \sum_{l=1}^{n} (X_l b_l)\bigg) + \bigg(h_{n+1,n+1} b_{n+1} z + H \sum_{l=1}^{n} (Z_l b_{n+1,n+1}^l)\bigg)$$

and

$$g_{22}(x,z) = 2h_{n+1,n+1}(b_{n+1}^T x + b_{n+1,n+1}^{n+1} z).$$

Then the total derivative takes the form

$$\begin{aligned}\dot V(x,z) = x^T(A^T H + HA)x &+ x^T \bigg\{ \sum_{l=1}^{n} \big((X_l B_l)^T H + H(X_l B_l)\big) \bigg\} x \\ &+ x^T \bigg\{ \sum_{l=1}^{n} \big((Z_l b_l^T)^T H + H(Z_l b_l^T)\big) \bigg\} x \\ &+ 2x^T \bigg(h_{n+1,n+1} B_{n+1} x + H \sum_{l=1}^{n} (X_l b_l) \bigg) z \\ &+ 2x^T \bigg(h_{n+1,n+1} b_{n+1} z + H \sum_{l=1}^{n} (Z_l b_{n+1,n+1}^l) \bigg) z \\ &+ 2h_{n+1,n+1}(b_{n+1}^T x + b_{n+1,n+1}^{n+1} z) z^2.\end{aligned}$$

Let us consider some addends of this expression separately.

1. Symmetric matrix $C = -A^T H - HA$ is, in accordance with conditions of Theorem 4.1, positive definite.

2. Let us denote

$$X = \begin{pmatrix} X_1^T \\ X_2^T \\ \dots \\ X_n^T \end{pmatrix}.$$

Then, by using (7), we transform the expression in the second addend:

$$\sum_{l=1}^{n} \big((X_l B_l)^T H + H(X_l B_l)\big) = (\bar B^T X)^T H + H(\bar B^T X).$$

3. With the aid of (6) we transform the expression in the third addend:

$$\sum_{l=1}^{n} Z_l b_l^T = z\tilde B_{n+1}.$$

4. For the fourth term we get

$$2x^T \bigg(h_{n+1,n+1} B_{n+1} x + H \sum_{l=1}^{n} (X_l b_l) \bigg) z = 2zx^T \big(h_{n+1,n+1} B_{n+1} + H\tilde B_{n+1} \big) x.$$

5. The fifth addend turns into

$$2x^T\Bigg(h_{n+1,n+1}b_{n+1}z + H\sum_{l=1}^{n}(Z_l b^l_{n+1,n+1})\Bigg)z = 2x^T\big(h_{n+1,n+1}b_{n+1} + H\tilde{b}_{n+1}\big)z^2.$$

After above transformations the total derivative is simplified as

$$\begin{aligned}\dot{V}(x,z) = & -x^T Cx + x^T\left((\bar{B}^T X)^T H + H(\bar{B}^T X)\right)x + zx^T\left(\tilde{B}^T_{n+1}H + H\tilde{B}_{n+1}\right)x\\ & + zx^T\big(2h_{n+1,n+1}B_{n+1} + (\tilde{B}^T_{n+1}H + H\tilde{B}_{n+1})\big)x\\ & + 2x^T\big(2h_{n+1,n+1}b_{n+1} + H\tilde{b}_{n+1}\big)z^2 + 2h_{n+1,n+1}b^{n+1}_{n+1,n+1}z^3\end{aligned}$$

and, finally, if we take into account (8) (since obviously it can be put $h_{n+1,n+1} = 1$) and condition $b^{n+1}_{n+1,n+1} = 0$, it becomes

$$\dot{V}(x,z) = -x^T\big\{C - \big((\bar{B}^T X)^T H + H(\bar{B}^T X)\big) - zR(H)\big\}x. \tag{9}$$

Estimation of (9) gives (we take into account the property $\|X\| = \|x\|$)

$$\begin{aligned}& -x^T\left\{C - \big((\bar{B}^T X)^T H + H(\bar{B}^T X)\big) - zR(H)\right\}x\\ & - x^T Cx + x^T\left\{\big((\bar{B}^T X)^T H + H(\bar{B}^T X)\big) - zR(H)\right\}x\\ \le & -\lambda_{\min}(C)\|x\|^2 + \lambda_{\max}\left\{\big((\bar{B}^T X)^T H + H(\bar{B}^T X)\big) - zR(H)\right\}\|x\|^2\\ \le & \big(-\lambda_{\min}(C) + 2\|H\tilde{B}^T\|\cdot\|x\| + \|R(H)\|\cdot\|z\|\big)\|x\|^2.\end{aligned}$$

Then for stability of the system (1) it is sufficient that

$$\lambda_{\min}(C) - 2\|H\bar{B}^T\|\cdot\|x\| - \|R(H)\|\cdot\|z\| \ge 0. \tag{10}$$

Since

$$V(x,z) = x^T Hx + z^2 \le \lambda_{\max}(H)\|x\|^2 + \|z\|^2, \tag{11}$$

as it follows from (10) and (11), a guaranteed zone of stability has the form

$$x^T Hx + z^2 \le \alpha$$

with

$$\alpha = \frac{\lambda_{\max}(H)\cdot(\lambda_{\min}(C))^2}{\lambda_{\max}(H)\|R(H)\|^2 + 4\|H\bar{B}^T\|^2}.$$

The theorem is proved.

The assertion (formulated in Remark 4.1) concerning the maximal interval of existence follows, obviously, from the method of the proof which uses the Liapunov function.

Corollary 4.1 *From the proof of Theorem 4.1 and from representation (9) the following corollary follows: Let all assumptions of Theorem 4.1 be valid and, moreover,*

$$H\bar{B}^T = 0, \qquad R(H) = 0.$$

Then the trivial solution of the system (1) is globally stable. In this case for the global stability it is sufficient if the matrix $C = -A^T H - HA$ *is only positive semi-definite.*

5 Example

Let us consider system (1) for $n = 2$, i.e. the system of three equations

$$\begin{aligned}
\dot{x} &= -x + ay + b_{11}^1 x^2 + b_{22}^1 y^2 + b_{33}^1 z^2 + 2b_{12}^1 xy + 2b_{13}^1 xz + 2b_{23}^1 yz,\\
\dot{y} &= \qquad\; -y + b_{11}^2 x^2 + b_{22}^2 y^2 + b_{33}^2 z^2 + 2b_{12}^2 xy + 2b_{13}^2 xz + 2b_{23}^2 yz,\\
\dot{z} &= \qquad\qquad\quad b_{11}^3 x^2 + b_{22}^3 y^2 + b_{33}^3 z^2 + 2b_{12}^3 xy + 2b_{13}^3 xz + 2b_{23}^3 yz,
\end{aligned} \tag{12}$$

where $a, b_{ij}^k,\ i, j, k = 1, 2, 3,\ i \le j$ are constants, $a > 0$ and $b_{33}^2 \ne 0$. As it follows from Theorem 4.1, the stability of trivial solution of system (12) will be proved if there exists a symmetric positive definite matrix

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

such that the matrix $C = -A^T H - HA$ is positive definite too and, except for

$$b_{33}^3 = 0, \quad H\tilde{b}_3 + 2b_3 = 0 \tag{13}$$

with $b_3^T = (b_{13}^3, b_{23}^3)$ and $\tilde{b}_3^T = (b_{33}^1, b_{33}^2)$. It is easy to see that

$$C = \begin{pmatrix} 2h_{11} & 2h_{12} - ah_{11} \\ 2h_{12} - ah_{11} & 2(h_{22} - ah_{12}) \end{pmatrix}.$$

Combining conditions for positive definiteness of matrix C and conditions (13) we get (note that from the existence of positive definite matrix C follows the existence of positive definite matrix H – see end of the Section 2):

$$\begin{cases} h_{11} > 0, \\ 4h_{11}(h_{22} - ah_{12}) - (2h_{12} - ah_{11})^2 > 0, \\ h_{11} b_{33}^1 + h_{12} b_{33}^2 = -2b_{13}^3, \\ h_{12} b_{33}^1 + h_{22} b_{33}^2 = -2b_{23}^3. \end{cases} \tag{14}$$

The last two equations yield

$$\begin{aligned}
h_{12} &= -\frac{1}{b_{33}^2}\,(h_{11} b_{33}^1 + 2b_{13}^3),\\
h_{22} &= \frac{1}{(b_{33}^2)^2}\,[(b_{33}^1)^2 h_{11} + 2(b_{33}^1 b_{13}^3 - b_{23}^3 b_{33}^2)].
\end{aligned} \tag{15}$$

With the aid of (15) and the second inequality in (14) we obtain:

$$a^2 h_{11}^2 + \frac{8}{(b_{33}^2)^2}\,(b_{33}^1 b_{13}^3 + b_{23}^3 b_{33}^2) h_{11} + 16\left(\frac{b_{13}^3}{b_{33}^2}\right)^2 < 0.$$

Then, for $h_{11} > 0$,

$$b_3^T \tilde{b}_3 = b_{13}^3 b_{33}^1 + b_{23}^3 b_{33}^2 < -a|b_{13}^3 b_{33}^2|$$

is necessary and sufficient. Using the vectors b_3 and $\tilde{b}_3$ we see that h_{11} can vary within the interval

$$\begin{aligned} -\frac{4}{a^2}\cdot\left[\frac{b_3^T\tilde{b}_3}{(b_{33}^2)^2} + \sqrt{\left[\frac{b_3^T\tilde{b}_3}{(b_{33}^2)^2}\right]^2 - a^2\left(\frac{b_{13}^3}{b_{33}^2}\right)^2}\,\right] &< h_{11} \\ < -\frac{4}{a^2}\cdot\left[\frac{b_3^T\tilde{b}_3}{(b_{33}^2)^2} - \sqrt{\left[\frac{b_3^T\tilde{b}_3}{(b_{33}^2)^2}\right]^2 - a^2\left(\frac{b_{13}^3}{b_{33}^2}\right)^2}\,\right]&. \end{aligned} \tag{16}$$

So, for stability of the trivial solution of system (12) the following conditions are sufficient

$$\begin{aligned} &b_{33}^3 = 0, \\ &b_3^T\tilde{b}_3 < -a|b_{13}^3 b_{33}^2|. \end{aligned} \tag{17}$$

Remark 5.1 The first two inequalities in (14) define a convex cone in the space (h_{11}, h_{12}, h_{22}) with the vertex at the origin. Next two equations define a straight line in this space. Consequently, geometrically relations (14) express the conditions of an intersection of a straight line and a cone.

Let us consider a partial case of the system (12) when $B_1 = B_2 = B_3 = \tilde{B}_3 = 0$ and $b_{33}^3 = 0$. Then $H\bar{B}^T = 0$, $R(H) = 0$ and the system has the form

$$\begin{aligned} \dot{x} &= -x + ay + b_{33}^1 z^2, \\ \dot{y} &= \qquad\; -y + b_{33}^2 z^2, \\ \dot{z} &= \qquad\qquad 2b_{13}^3 xz + 2b_{23}^3 yz. \end{aligned} \tag{18}$$

Suppose that b_{33}^1, b_{33}^2, b_{13}^3 and b_{23}^3 satisfy conditions (17). Let h_{11} be taken in accordance with inequalities (16) and compute h_{12} and h_{22} by formulas (15). Then the corresponding Liapunov function has the form

$$V(x, y, z) = h_{11}x^2 + 2h_{12}xy + h_{22}y^2 + z^2$$

and its derivative is

$$\dot{V}(x, y, z) = -2\left\{h_{11}x^2 + 2\left(h_{12} - \frac{1}{2}\,ah_{11}\right)xy + (h_{22} - ah_{12})y^2\right\}.$$

This derivative is negative semi-definite in $\mathbb{R}^3$, i.e. the trivial solution of (18) is globally stable. This is in accordance with Corollary 4.1.

Stationary points of (18) are solutions of the system

$$\begin{aligned} -x + ay + \qquad\qquad b^1_{33}z^2 &= 0,\\ -y + \qquad\qquad b^2_{33}z^2 &= 0,\\ 2b^3_{13}xz + 2b^3_{23}yz &= 0. \end{aligned}$$

Seeking for the stationary point of this system we see that, in view of the inequality in (17), the stationary point is the only one – namely, the origin of coordinates.

Let us consider some of possible *limiting* cases.

1. Suppose that instead of inequality the equality in (17) holds and, moreover,

$$b_3^T\tilde{b}_3 = b^3_{13}b^1_{33} + b^3_{23}b^2_{33} = -ab^3_{13}b^2_{33} \quad \text{and} \quad b^3_{13}b^2_{33} > 0. \tag{19}$$

Then there exists a set of stationary points which lies on the curve

$$x = (ab^2_{33} + b^1_{33})z^2, \quad y = b^2_{33}z^2, \quad -\infty < z < \infty.$$

In accordance with (16) we put (as a limiting case)

$$h_{11} = -\frac{4}{a^2}\cdot\frac{b_3^T\tilde{b}_3}{(b^2_{33})^2} = \frac{4}{a}\cdot\frac{b^3_{13}}{b^2_{33}}.$$

Then formulas (15) give

$$\begin{aligned} h_{12} &= -\frac{2b^3_{13}}{b^2_{33}}\left(\frac{2}{a}\cdot\frac{b^1_{33}}{b^2_{33}} + 1\right),\\ h_{22} &= \frac{2}{(b^2_{33})^2}\left(\frac{2}{a}\cdot\frac{b^3_{13}(b^1_{33})^2}{b^2_{33}} + b^1_{33}b^3_{13} - b^3_{23}b^2_{33}\right). \end{aligned} \tag{20}$$

Conditions of positivity definiteness of the Liapunov function have the form

$$h_{11} > 0, \quad h_{11}h_{22} - h^2_{12} > 0. \tag{21}$$

If we take into account (20), these inequalities take the form

$$b^3_{13}b^2_{33} > 0, \quad \frac{2}{a}\cdot\frac{b^3_{13}}{(b^2_{33})^3}\cdot(b^3_{13}b^1_{33} + b^3_{23}b^2_{33}) + \frac{(b^3_{13})^2}{(b^2_{33})^2} < 0.$$

If (19) holds, then the second inequality turns into

$$-\frac{(b^3_{13})^2}{(b^2_{33})^2} < 0$$

and always holds. So, if (19) holds, the trivial solution of the system (18) is globally stable.

2. Suppose that instead of inequality the equality in (17) holds and, moreover,

$$b_3^T\tilde{b}_3 = b^3_{13}b^1_{33} + b^3_{23}b^2_{33} = ab^3_{13}b^2_{33} \quad \text{and} \quad b^3_{13}b^2_{33} < 0. \tag{22}$$

Then only the origin is a stationary point. Let us put (in accordance with (16) in a limiting case)

$$h_{11} = -\frac{4}{a^2}\cdot\frac{b_3^T\tilde{b}_3}{(b_{33}^2)^2} = -\frac{4}{a^2}\cdot\frac{b_{13}^3}{b_{33}^2}.$$

Then formulas (15) give

$$\begin{aligned} h_{12} &= -\frac{2b_{13}^3}{b_{33}^2}\Big(-\frac{2}{a}\cdot\frac{b_{33}^1}{b_{33}^2}+1\Big),\\ h_{22} &= \frac{2}{(b_{33}^2)^2}\Big(-\frac{2}{a}\cdot\frac{b_{13}^3(b_{33}^1)^2}{b_{33}^2}+b_{33}^1b_{13}^3-b_{23}^3b_{33}^2\Big). \end{aligned} \tag{23}$$

Conditions (21) for positivity definiteness of the Liapunov function take (in view of (23)) the form

$$b_{13}^3b_{33}^2<0,\qquad \frac{2}{a}\cdot\frac{b_{13}^3}{(b_{33}^2)^3}\cdot(b_{13}^3b_{33}^1+b_{23}^3b_{33}^2)-\frac{(b_{13}^3)^2}{(b_{33}^2)^2}>0.$$

If (22) holds, then the second inequality turns into

$$\frac{(b_{13}^3)^2}{(b_{33}^2)^2}>0$$

and always holds too. So, if (22) holds, then the trivial solution of the system (18) is globally stable too.

Remark 5.2 In the above considered particular limiting cases **1** and **2** the condition (14) can be geometrically interpreted as a contact of a straight line with a convex cone.

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